

EQUALITY IN BORELL-BRASCAMP-LIEB INEQUALITIES ON CURVED SPACES

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ABSTRACT. By using optimal mass transportation and a quantitative Hölder inequality, we provide estimates for the Borell-Brascamp-Lieb deficit on complete Riemannian manifolds. As a first consequence, equality cases in Borell-Brascamp-Lieb inequalities (including Brunn-Minkowski and Prékopa-Leindler inequalities) are characterized in terms of the optimal transport map between suitable marginal probability measures. When the n -dimensional Riemannian manifold has Ricci curvature $\text{Ric}(M) \geq (n-1)k$ for some $k \in \mathbb{R}$, equality in the Borell-Brascamp-Lieb inequality is expected only when a particular region of the manifold between the marginal supports has constant sectional curvature k . In particular, a full picture is provided concerning the equality in the Lott-Sturm-Villani-type distorted Brunn-Minkowski inequality on the hyperbolic space and sphere. Another byproduct of the Borell-Brascamp-Lieb deficit estimate is a weak stability result for the log-Brunn-Minkowski inequality. Some related results for (not necessarily reversible) Finsler manifolds are also presented.

Keywords: Borell-Brascamp-Lieb inequality; Brunn-Minkowski inequality; Prékopa-Leindler inequality; equality case; optimal mass transportation; Riemannian manifold; Finsler manifold.

MSC: 49Q20; 53C21; 39B62; 53C24; 58E35.

1. INTRODUCTION

1.1. Background and motivation. The Borell-Brascamp-Lieb inequality in the Euclidean space \mathbb{R}^n states that for every fixed $s \in (0, 1)$, $p \geq -\frac{1}{n}$ and integrable functions $f, g, h : \mathbb{R}^n \rightarrow [0, \infty)$ which satisfy

$$h((1-s)x + sy) \geq \mathcal{M}_s^p(f(x), g(y)) \quad \text{for all } x, y \in \mathbb{R}^n, \quad (1.1)$$

one has

$$\int_{\mathbb{R}^n} h \geq \mathcal{M}_s^{\frac{p}{1+np}} \left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g \right). \quad (1.2)$$

Here, for every $s \in (0, 1)$, $p \in \mathbb{R} \cup \{\pm\infty\}$ and $a, b \geq 0$, the p -mean is defined by

$$\mathcal{M}_s^p(a, b) = \begin{cases} ((1-s)a^p + sb^p)^{1/p} & \text{if } ab \neq 0, \\ 0 & \text{if } ab = 0, \end{cases}$$

with the conventions $\mathcal{M}_s^{-\infty}(a, b) = \min\{a, b\}$; $\mathcal{M}_s^0(a, b) = a^{1-s}b^s$; and $\mathcal{M}_s^{+\infty}(a, b) = \max\{a, b\}$ if $ab \neq 0$ and $\mathcal{M}_s^{+\infty}(a, b) = 0$ if $ab = 0$.

One of the most important consequences of the Borell-Brascamp-Lieb inequality (for $p = +\infty$ and indicator functions) is the usual Brunn-Minkowski inequality which relates the volume of two convex bodies A and B in \mathbb{R}^n with the volume of their Minkowski sum $(1-s)A + sB = \{(1-s)x + sy : x \in A, y \in B\}$ as

$$V((1-s)A + sB)^{\frac{1}{n}} \geq (1-s)V(A)^{\frac{1}{n}} + sV(B)^{\frac{1}{n}}. \quad (1.3)$$

An equivalent form of (1.3), coming also from the Borell-Brascamp-Lieb inequality (for $p = 0$ and indicator functions), is the log-Brunn-Minkowski inequality, – or the geometric form of the Prékopa-Leindler inequality, – which states that

$$V((1-s)A + sB) \geq V(A)^{1-s}V(B)^s. \quad (1.4)$$

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The *equality* and *stability* issues in the aforementioned inequalities (1.2)-(1.4) constitute a continuous source of inspiration for further investigations. After the pioneering works by Brunn and Minkowski, it is well known for more than a century that equality in (1.3) holds if and only if the sets A and B are homothetic, while equality in (1.4) holds if and only if the sets A and B are translates. The equality case in the generic Borell-Brascamp-Lieb inequality (1.2) has been studied in the mid of seventies by Dubuc [15] on \mathbb{R}^n by using certain convexity results together with a careful inductive argument w.r.t. the dimension of the space \mathbb{R}^n . Later on, Dancs and Uhrin [13, 14] obtained some qualitative Borell-Brascamp-Lieb inequalities on \mathbb{R} , providing also some higher-dimensional versions. A few years ago, Ball and Böröczky [2, 3] obtained stability results for the one-dimensional functional Prékopa-Leindler inequality with some extensions also to higher-dimensions. Very recently, various stability results are established in \mathbb{R}^n for the generic Borell-Brascamp-Lieb inequality by Ghilli and Salani [19], Rossi and Salani [29], for the Prékopa-Leindler inequality by Bucur and Fragalà [7], and for the Brunn-Minkowski inequality by Christ [9] and Figalli and Jerison [16, 17]. The common strategy in the aforementioned papers is the use of various arguments from convex analysis combined usually with some inductive step w.r.t. the dimension, by fully exploring the Euclidean character of the space.

As far as we know, no equality/stability results are available for Borell-Brascamp-Lieb inequalities on *curved spaces*. It is clear that the arguments from the aforementioned papers (see [2], [3], [7], [9], [13], [15], [16], [17], and references therein) cannot be applied in such a nonlinear setting.

The starting point of our investigation is the celebrated work by Cordero-Erausquin, McCann and Schmuckenschläger [12] who established a Riemannian version of the Borell-Brascamp-Lieb inequality via optimal mass transportation culminating in a distorted Jacobian determinant inequality. The Finslerian counterparts of the results from [12] are provided by Ohta [26]. We point out that the first optimal mass transportation approach to Prékopa-Leindler, Brunn-Minkowski and Brascamp-Lieb inequalities has been developed by McCann [24], [25, Appendix D] in the Euclidean setting.

The main purpose of our paper is to characterize the equality in Borell-Brascamp-Lieb inequalities on complete n -dimensional Riemannian/Finsler manifolds for the whole spectrum of the parameter $p \geq -\frac{1}{n}$ by exploring a quantitative Hölder inequality and the theory of optimal mass transportation. Although our approach is more appropriate for characterizing equality cases, we furnish also some rigidity and weak stability results for geometric inequalities (e.g. for log-Brunn-Minkowski inequality).

In the sequel, we present some of our achievements.

1.2. Main results. In order to present our results, let (M, w) be a complete n -dimensional Riemannian manifold ($n \geq 2$) with its natural metric $d : M \times M \rightarrow [0, \infty)$. For a fixed $s \in (0, 1)$ and $(x, y) \in M \times M$ let

$$Z_s(x, y) = \{z \in M : d(x, z) = sd(x, y), d(z, y) = (1 - s)d(x, y)\}$$

be the set of s -intermediate points between x and y , replacing the convex combination in (1.1). Since (M, d) is complete, $Z_s(x, y) \neq \emptyset$ for every $x, y \in M$. Accordingly, the set

$$Z_s(A, B) = \bigcup_{(x, y) \in A \times B} Z_s(x, y)$$

replaces the Minkowski sum of the nonempty sets $A, B \subset M$.

Let $s \in (0, 1)$ and $p \geq -\frac{1}{n}$. If $f, g, h : M \rightarrow [0, \infty)$ are three non-zero, compactly supported integrable functions, the natural Riemannian reformulation of (1.1) reads as

$$h(z) \geq \mathcal{M}_s^p \left(\frac{f(x)}{v_{1-s}(y, x)}, \frac{g(y)}{v_s(x, y)} \right) \quad \text{for all } (x, y) \in M \times M, z \in Z_s(x, y), \quad (1.5)$$

where v_s is the volume distortion coefficient (see (3.1) for its precise definition). Under the assumption (1.5), the main result of Cordero-Erausquin, McCann and Schmuckenschläger [12] reads as

$$\int_M h \geq \mathcal{M}_s^{\frac{p}{1+np}} \left(\int_M f, \int_M g \right),$$

where the integrals are considered w.r.t. the canonical volume element dV_w on (M, w) .

For simplicity of notation, let $\|\cdot\|_1$ be the L^1 -norm of any integrable function on M . For the above functions f, g and h , let us consider the *Borell-Brascamp-Lieb deficit* given by

$$\delta_{M,s}^p(f, g, h) = \frac{\|h\|_1}{\mathcal{M}_s^{\frac{p}{1+p^n}}(\|f\|_1, \|g\|_1)} - 1.$$

We first provide an estimate for the Borell-Brascamp-Lieb deficit on a generic Riemannian manifold which will be achieved by using the theory of optimal mass transportation and a quantitative Hölder inequality:

Theorem 1.1. (Estimate of the Borell-Brascamp-Lieb deficit) *Let (M, w) be a complete n -dimensional Riemannian manifold, $s \in (0, 1)$, $p \geq -\frac{1}{n}$ and $f, g, h : M \rightarrow [0, \infty)$ be three non-zero, compactly supported integrable functions satisfying (1.5). Then*

$$\delta_{M,s}^p(f, g, h) \geq \int_M \tilde{f}(x) G_s^{p,n} \left(\frac{f(x)}{v_{1-s}(\psi(x), x)}, \frac{g(\psi(x))}{v_s(x, \psi(x))}, \frac{1}{\|f\|_1}, \frac{1}{\|g\|_1} \right) dV_w,$$

where $\psi : M \rightarrow M$ is the unique optimal transport map from the measure $\mu = \tilde{f}dV_w$ to $\nu = \tilde{g}dV_w$ with densities $\tilde{f} = f/\|f\|_1$, $\tilde{g} = g/\|g\|_1$, and $G_s^{p,n} \geq 0$ is the gap-function given in Lemma 2.1.

The uniqueness of the optimal transport map $\psi : M \rightarrow M$ from the probability measure $\mu = \tilde{f}dV_w$ to $\nu = \tilde{g}dV_w$ is well known by McCann [23] having the form $\psi(x) = \exp_x(-\nabla\varphi(x))$ for a.e. $x \in \text{supp } f$ for some $d^2/2$ -concave function $\varphi : M \rightarrow \mathbb{R}$, where ∇ denotes the Riemannian gradient. Let $\psi_s : M \rightarrow M$ be the s -interpolant optimal transport map $\psi_s(x) = \exp_x(-s\nabla\varphi(x))$ for a.e. $x \in \text{supp } f$, and $\text{Jac}(\psi_s)(x)$ its Jacobian in a.e. $x \in \text{supp } f$, see (3.2).

By Theorem 1.1 the equality in the Borell-Brascamp-Lieb inequality can be characterized by studying the properties of the gap-function $G_s^{p,n}$, leading us to the following result:

Theorem 1.2. (Equality in Borell-Brascamp-Lieb inequality; $p > -\frac{1}{n}$) *Let (M, w) be a complete n -dimensional Riemannian manifold, $s \in (0, 1)$, $p > -\frac{1}{n}$ and $f, g, h : M \rightarrow [0, \infty)$ be three non-zero, compactly supported integrable functions satisfying (1.5). Then the following two assertions are equivalent:*

- (a) $\delta_{M,s}^p(f, g, h) = 0$, i.e., there exists equality in the Borell-Brascamp-Lieb inequality;
- (b) the following statements simultaneously hold:
 - (i) $\text{supp } h = \psi_s(\text{supp } f)$ up to a null measure set;
 - (ii) $\text{Jac}(\psi_s)(x) = v_{1-s}(\psi(x), x) \left[\mathcal{M}_s^{\frac{p}{pn+1}} \left(1, \frac{\|g\|_1}{\|f\|_1} \right) \right]^{\frac{pn}{pn+1}}$ for a.e. $x \in \text{supp } f$;
 - (iii) for a.e. $x \in \text{supp } f$, one has

$$\frac{h(\psi_s(x))}{\left[\mathcal{M}_s^{\frac{p}{pn+1}}(\|f\|_1, \|g\|_1) \right]^{\frac{1}{pn+1}}} = \frac{f(x)}{v_{1-s}(\psi(x), x) \|f\|_1^{\frac{1}{pn+1}}} = \frac{g(\psi(x))}{v_s(x, \psi(x)) \|g\|_1^{\frac{1}{pn+1}}}.$$

The equality in the Borell-Brascamp-Lieb inequality for $p = -\frac{1}{n}$ is treated separately in Theorem 3.1.

We notice that usually the inclusion $\psi_s(\text{supp } f) \subset \text{supp } h$ is strict. According to Theorem 1.2(b), the equality in the Borell-Brascamp-Lieb inequality implies the equality $\psi_s(\text{supp } f) = \text{supp } h$, which corresponds to the Alesker-Dar-Milman relation concerning the parametrization of the Minkowski sum of two sets in \mathbb{R}^n ; for further details, see Remark 4.3.

Theorem 1.2 provides important consequences; in the sequel, we shall present roughly three of them:

I. *Equality in Borell-Brascamp-Lieb inequality implies constant curvature.* We state that the equality in the Borell-Brascamp-Lieb inequality on a Riemannian manifold with Ricci curvature bounded below can be expected to hold only when a particular region of the manifold between the marginal supports has constant sectional curvature; the precise statement can be found in Theorem 4.1.

II. *Equality in distorted Brunn-Minkowski inequality à la Lott-Sturm-Villani.* For $s \in (0, 1)$, $k \in \mathbb{R}$ and $n \geq 2$, let us recall the distortion coefficient

$$\tau_s^{k,n}(\theta) = \begin{cases} s^{\frac{1}{n}} \left(\sinh \left(\sqrt{-\frac{k}{n-1}} s \theta \right) / \sinh \left(\sqrt{-\frac{k}{n-1}} \theta \right) \right)^{1-\frac{1}{n}} & \text{if } k\theta^2 < 0; \\ s & \text{if } k\theta^2 = 0; \\ s^{\frac{1}{n}} \left(\sin \left(\sqrt{\frac{k}{n-1}} s \theta \right) / \sin \left(\sqrt{\frac{k}{n-1}} \theta \right) \right)^{1-\frac{1}{n}} & \text{if } 0 < k\theta^2 < (n-1)\pi^2; \\ +\infty & \text{if } k\theta^2 \geq (n-1)\pi^2, \end{cases}$$

introduced independently by Lott and Villani [21] and Sturm [31]. The number $\tau_s^{k,n}$ encodes information on the curvature of space forms serving as model structures in the definition of the Lott-Sturm-Villani's curvature-dimension condition $\text{CD}(k, n)$ on metric measure spaces.

Let (M, w) be a complete n -dimensional Riemannian manifold with Ricci curvature bounded below, i.e., $\text{Ric}(M) \geq k(n-1)$ for some $k \in \mathbb{R}$ (which is equivalent to the validity of $\text{CD}(k, n)$, see e.g. Sturm [31, Theorem 1.7]). Let us denote by \mathbf{m} the canonical measure on M w.r.t. the volume element dV_w . By the theory of Lott-Sturm-Villani one has the distorted Brunn-Minkowski inequality

$$\mathbf{m}(Z_s(A, B))^{\frac{1}{n}} \geq \tau_{1-s}^{k,n}(\Theta_{A,B}) \mathbf{m}(A)^{\frac{1}{n}} + \tau_s^{k,n}(\Theta_{A,B}) \mathbf{m}(B)^{\frac{1}{n}}, \quad (1.6)$$

where $A, B \subset M$ are measurable sets with $\mathbf{m}(A) \neq 0 \neq \mathbf{m}(B)$ and

$$\Theta_{A,B} = \begin{cases} \inf_{(x,y) \in A \times B} d(x, y) & \text{if } k \geq 0; \\ \sup_{(x,y) \in A \times B} d(x, y) & \text{if } k < 0, \end{cases} \quad (1.7)$$

see e.g. Sturm [31, Proposition 2.1] and Villani [32, Theorem 18.5]. Theorem 1.2 provides the following scenario concerning the equality in the Brunn-Minkowski inequality (1.6):

Theorem 1.3. (Equality in distorted Brunn-Minkowski inequality) *Let (M, w) be a complete n -dimensional Riemannian manifold, $A, B \subset M$ be compact sets with $\mathbf{m}(A) \neq 0 \neq \mathbf{m}(B)$ and $s \in (0, 1)$. Then the following statements hold:*

- (i) (Positively curved case) *If $\text{Ric}(M) \geq (n-1)k$ for some $k > 0$, equality holds in (1.6) if and only if $Z_s(A, B) = A = B$ up to a null measure set;*
- (ii) (Negatively curved case) *If (M, g) has nonpositive, nonzero sectional curvature and $\text{Ric}(M) \geq (n-1)k$ for some $k < 0$, equality cannot hold in (1.6);*
- (iii) (Null curved case) *If $\text{Ric}(M) \geq 0$ and equality holds in (1.6) then (M, w) is a.e. flat between A and B in the sense that for a.e. $x \in A$ the sectional curvature vanishes along the geodesics $t \mapsto \psi_t(x)$, $t \in [0, 1]$, where $t \mapsto \psi_t$ denotes the optimal transport map from the measure $\mu = \mathbb{1}_A / \mathbf{m}(A) dV_w$ to $\nu = \mathbb{1}_B / \mathbf{m}(B) dV_w$. (The latter conclusion is relevant whenever $A \neq B$ up to a null measure set; otherwise, the optimal transport map is the identity, $\Theta_{A,B} = 0$, and equality in (1.6) trivially holds if and only if $Z_s(A, B) = A = B$ up to a null measure set).*

In particular, by Theorem 1.3 we have that the equality in (1.6):

- (a) is characterized by the overlapping of the two sets A and B on the standard unit sphere \mathbb{S}^n ;
- (b) cannot hold for any positively measured sets A and B on the hyperbolic space \mathbb{H}^n ;
- (c) is characterized by the homothetic position of the convex sets A and B in \mathbb{R}^n .

III. *Equality in the Euclidean Brunn-Minkowski inequality via Caffarelli's regularity result.* Theorem 1.2 provides a novel, optimal transport based proof of the latter characterization (c), i.e., the equality in the Euclidean Brunn-Minkowski inequality (1.3) by exploring the celebrated regularity result of Caffarelli [8], see Corollary 4.2.

As we already pointed out, our approach is efficient to state equalities in Borell-Brascamp-Lieb inequalities rather than to establish stability results. However, some weak stability results can still be obtained. For simplicity, we shall state a quantitative result for the log-Brunn-Minkowski inequality (1.4) in \mathbb{R}^n as an application of Theorem 1.1 (a slightly more general form is given in Proposition 3.1); an exhaustive study of this inequality can be found in Böröczky, Lutwak, Yang and Zhang [6].

Corollary 1.1. (Quantitative log-Brunn-Minkowski inequality) *Let $n \geq 2$ and $s \in (0, 1)$. For every nonempty compact sets $A, B \subset \mathbb{R}^n$ with $V(A) \neq 0 \neq V(B)$ we have*

$$\frac{V((1-s)A + sB)}{V(A)^{1-s}V(B)^s} - 1 \geq n\tilde{s} \frac{\left|V(A)^{\frac{\tilde{s}}{n}} - V(B)^{\frac{\tilde{s}}{n}}\right|^{\frac{1}{\tilde{s}}}}{(1-s)V(A)^{\frac{1}{n}} + sV(B)^{\frac{1}{n}}},$$

where $\tilde{s} = \min(s, 1-s)$.

The organization of the paper is as follows. In Section 2 we provide the proof of the quantitative Hölder inequality which is crucial in the proof of Theorem 1.1. In Section 3 we prove simultaneously Theorems 1.1 and 1.2; as a consequence we prove Corollary 1.1 and its slightly more general form. Section 4 is devoted to rigidities, i.e., we present results in specific geometric settings where the equality in Borell-Brascamp-Lieb inequalities implies restrictions on the curvature and structure of the involved functions/sets. Accordingly, in §4.1 we deal with Riemannian manifolds, by proving Theorems 4.1 and 1.3, and Corollary 4.2, respectively. In §4.2 certain Borell-Brascamp-Lieb inequalities are presented on not necessarily reversible Finsler manifolds; here, some unusual phenomena are described within the class of Minkowski spaces.

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2. A QUANTITATIVE HÖLDER INEQUALITY

According to Gardner [18, Lemma 10.1], one has the Hölder inequality

$$\mathcal{M}_s^p(a, b)\mathcal{M}_s^q(c, d) \geq \mathcal{M}_s^\eta(ac, bd), \quad (2.1)$$

for every $a, b, c, d \geq 0, s \in (0, 1)$ and $p, q \in \mathbb{R}$ such that $p + q \geq 0$ with $\eta = \frac{pq}{p+q}$ when p and q are not both zero, and $\eta = 0$ if $p = q = 0$.

In the sequel, we provide a technical improvement of (2.1) needed to prove Theorem 1.1.

Lemma 2.1. (Quantitative Hölder inequality) *Let $n \in \mathbb{N} \setminus \{0\}$, $s \in (0, 1)$ and $a, b, c, d > 0$ be arbitrarily fixed numbers. For a fixed $p > -\frac{1}{n}$ let $\tilde{p} = \frac{p}{pn+1}$.*

(i) *If $p \in (0, \infty)$, then*

$$\mathcal{M}_s^p(a, b)\mathcal{M}_s^{-\tilde{p}}(c, d) \geq \mathcal{M}_s^{-\frac{1}{n}}(ac, bd) [1 + G_s^{p,n}(a, b, c, d)],$$

where

$$\begin{aligned} G_s^{p,n}(a, b, c, d) = & (1-s) \frac{n}{\max(pn, 1)} \left| \left[\mathcal{M}_s^{-p} \left(1, \frac{a}{b} \right) \right]^{\frac{p\tilde{p}n}{\max(pn, 1)}} - \left[\mathcal{M}_s^{-\frac{1}{n}} \left(1, \frac{bd}{ac} \right) \right]^{\frac{\tilde{p}}{\max(pn, 1)}} \right|^{\frac{\max(pn, 1)}{\tilde{p}n}} \\ & + s \frac{n}{\max(pn, 1)} \left| \left[\mathcal{M}_s^{-p} \left(\frac{b}{a}, 1 \right) \right]^{\frac{p\tilde{p}n}{\max(pn, 1)}} - \left[\mathcal{M}_s^{-\frac{1}{n}} \left(\frac{ac}{bd}, 1 \right) \right]^{\frac{\tilde{p}}{\max(pn, 1)}} \right|^{\frac{\max(pn, 1)}{\tilde{p}n}}. \end{aligned}$$

(ii) *If $p \in (-\frac{1}{n}, 0)$, then*

$$\mathcal{M}_s^p(a, b)\mathcal{M}_s^{-\tilde{p}}(c, d) \geq \mathcal{M}_s^{-\frac{1}{n}}(ac, bd) [1 + G_s^{p,n}(a, b, c, d)],$$

where

$$G_s^{p,n}(a, b, c, d) = G_s^{-\tilde{p},n}(c, d, a, b).$$

(iii) *If $p = +\infty$ (thus $\tilde{p} = \frac{1}{n}$), then*

$$\mathcal{M}_s^{+\infty}(a, b)\mathcal{M}_s^{-\frac{1}{n}}(c, d) \geq \mathcal{M}_s^{-\frac{1}{n}}(ac, bd) [1 + G_s^{+\infty,n}(a, b, c, d)],$$

where

$$G_s^{+\infty,n}(a, b, c, d) = n \min(s, 1-s) \frac{|a^{\frac{1}{n}} - b^{\frac{1}{n}}|}{(ab)^{\frac{1}{n}} [\mathcal{M}_s^{+\infty}(c, d)]^{\frac{1}{n}}} \left[\mathcal{M}_s^{-\frac{1}{n}}(ac, bd) \right]^{\frac{1}{n}}.$$

(iv) One has (for $p = 0$) that

$$\mathcal{M}_s^0(a, b)\mathcal{M}_s^0(c, d) = \mathcal{M}_s^0(ac, bd) \geq \mathcal{M}_s^{-\frac{1}{n}}(ac, bd) [1 + G_s^{0,n}(a, b, c, d)],$$

where

$$G_s^{0,n}(a, b, c, d) = n \min(s, 1-s) \left[\mathcal{M}_s^{\frac{1}{n}}(bd, ac) \right]^{-\frac{1}{n}} \left| (bd)^{\frac{\min(s, 1-s)}{n}} - (ac)^{\frac{\min(s, 1-s)}{n}} \right|^{\frac{1}{\min(s, 1-s)}}.$$

Proof. We first recall the quantitative Young inequality, i.e., if $r \geq 2$ and $\frac{1}{r} + \frac{1}{r'} = 1$, one has

$$uv \leq \frac{1}{r}u^r + \frac{1}{r'}v^{r'} - \frac{1}{r}|u - v^{\frac{1}{r-1}}|^r \quad \text{for every } u, v \geq 0, \quad (2.2)$$

see e.g. Cianchi [10].

(i) Let $p \in (0, \infty)$ and let us assume first that $pn \geq 1$. Applying inequality (2.2) for $r = \frac{p}{\bar{p}} = pn + 1 \geq 2$ and $r' = \frac{1}{\bar{p}}$, we have that

$$\begin{aligned} \frac{\left[\mathcal{M}_s^{\bar{p}}\left(\frac{1}{c}, \frac{1}{d}\right) \right]^{\bar{p}}}{\left[\mathcal{M}_s^p(a, b) \right]^{\bar{p}} \left[\mathcal{M}_s^{\frac{1}{n}}\left(\frac{1}{ac}, \frac{1}{bd}\right) \right]^{\bar{p}}} &= \frac{(1-s)^{\frac{1}{r}} a^{\bar{p}} (1-s)^{\frac{1}{r'}} \left(\frac{1}{ac}\right)^{\bar{p}}}{\left[\mathcal{M}_s^p(a, b) \right]^{\bar{p}} \left[\mathcal{M}_s^{\frac{1}{n}}\left(\frac{1}{ac}, \frac{1}{bd}\right) \right]^{\bar{p}}} + \frac{s^{\frac{1}{r}} b^{\bar{p}} s^{\frac{1}{r'}} \left(\frac{1}{bd}\right)^{\bar{p}}}{\left[\mathcal{M}_s^p(a, b) \right]^{\bar{p}} \left[\mathcal{M}_s^{\frac{1}{n}}\left(\frac{1}{ac}, \frac{1}{bd}\right) \right]^{\bar{p}}} \\ &\leq \frac{1}{r} \frac{(1-s) a^{\bar{p}r}}{\left[\mathcal{M}_s^p(a, b) \right]^{\bar{p}r}} + \frac{1}{r'} \frac{(1-s) \left(\frac{1}{ac}\right)^{\bar{p}r}}{\left[\mathcal{M}_s^{\frac{1}{n}}\left(\frac{1}{ac}, \frac{1}{bd}\right) \right]^{\bar{p}r}} \\ &\quad - \frac{1}{r} \left| \frac{(1-s)^{\frac{1}{r}} a^{\bar{p}}}{\left[\mathcal{M}_s^p(a, b) \right]^{\bar{p}}} - \left(\frac{(1-s)^{\frac{1}{r'}} \left(\frac{1}{ac}\right)^{\bar{p}}}{\left[\mathcal{M}_s^{\frac{1}{n}}\left(\frac{1}{ac}, \frac{1}{bd}\right) \right]^{\bar{p}}} \right)^{\frac{1}{r-1}} \right|^r \\ &\quad + \frac{1}{r} \frac{s b^{\bar{p}r}}{\left[\mathcal{M}_s^p(a, b) \right]^{\bar{p}r}} + \frac{1}{r'} \frac{s \left(\frac{1}{bd}\right)^{\bar{p}r}}{\left[\mathcal{M}_s^{\frac{1}{n}}\left(\frac{1}{ac}, \frac{1}{bd}\right) \right]^{\bar{p}r}} \\ &\quad - \frac{1}{r} \left| \frac{s^{\frac{1}{r}} b^{\bar{p}}}{\left[\mathcal{M}_s^p(a, b) \right]^{\bar{p}}} - \left(\frac{s^{\frac{1}{r'}} \left(\frac{1}{bd}\right)^{\bar{p}}}{\left[\mathcal{M}_s^{\frac{1}{n}}\left(\frac{1}{ac}, \frac{1}{bd}\right) \right]^{\bar{p}}} \right)^{\frac{1}{r-1}} \right|^r \\ &= 1 - \frac{1}{r} (1-s) \left| \left[\mathcal{M}_s^{-p} \left(1, \frac{a}{b}\right) \right]^{\bar{p}} - \left[\mathcal{M}_s^{-\frac{1}{n}} \left(1, \frac{bd}{ac}\right) \right]^{\frac{\bar{p}}{pn}} \right|^r \\ &\quad - \frac{1}{r} s \left| \left[\mathcal{M}_s^{-p} \left(\frac{b}{a}, 1\right) \right]^{\bar{p}} - \left[\mathcal{M}_s^{-\frac{1}{n}} \left(\frac{ac}{bd}, 1\right) \right]^{\frac{\bar{p}}{pn}} \right|^r. \end{aligned}$$

By rearranging the latter estimate, we obtain

$$\begin{aligned} \left[\mathcal{M}_s^p(a, b) \right]^{\bar{p}} \left[\mathcal{M}_s^{\frac{1}{n}}\left(\frac{1}{ac}, \frac{1}{bd}\right) \right]^{\bar{p}} &\geq \left[\mathcal{M}_s^{\bar{p}}\left(\frac{1}{c}, \frac{1}{d}\right) \right]^{\bar{p}} + \frac{1}{r} \left[\mathcal{M}_s^p(a, b) \right]^{\bar{p}} \left[\mathcal{M}_s^{\frac{1}{n}}\left(\frac{1}{ac}, \frac{1}{bd}\right) \right]^{\bar{p}} \times \\ &\quad \times \left[(1-s) \left| \left[\mathcal{M}_s^{-p} \left(1, \frac{a}{b}\right) \right]^{\bar{p}} - \left[\mathcal{M}_s^{-\frac{1}{n}} \left(1, \frac{bd}{ac}\right) \right]^{\frac{\bar{p}}{pn}} \right|^r + \right. \\ &\quad \left. + s \left| \left[\mathcal{M}_s^{-p} \left(\frac{b}{a}, 1\right) \right]^{\bar{p}} - \left[\mathcal{M}_s^{-\frac{1}{n}} \left(\frac{ac}{bd}, 1\right) \right]^{\frac{\bar{p}}{pn}} \right|^r \right]. \end{aligned}$$

Applying (2.1) for the right hand side, and using our notation $G_s^{p,n}(a, b, c, d)$, it follows that

$$[\mathcal{M}_s^p(a, b)]^{\tilde{p}} \left[\mathcal{M}_s^{\frac{1}{n}} \left(\frac{1}{ac}, \frac{1}{bd} \right) \right]^{\tilde{p}} \geq \left[\mathcal{M}_s^{\tilde{p}} \left(\frac{1}{c}, \frac{1}{d} \right) \right]^{\tilde{p}} \left(1 + \frac{p}{r} G_s^{p,n}(a, b, c, d) \right).$$

Note that $\frac{1}{\tilde{p}} = \frac{pn+1}{p} > 1$; then we may apply to the latter estimate Bernoulli's inequality, obtaining

$$\mathcal{M}_s^p(a, b) \mathcal{M}_s^{\frac{1}{n}} \left(\frac{1}{ac}, \frac{1}{bd} \right) \geq \mathcal{M}_s^{\tilde{p}} \left(\frac{1}{c}, \frac{1}{d} \right) \left(1 + \frac{p}{r\tilde{p}} G_s^{p,n}(a, b, c, d) \right).$$

Since $r\tilde{p} = p$ and

$$\mathcal{M}_s^{\tilde{p}} \left(\frac{1}{c}, \frac{1}{d} \right) = [\mathcal{M}_s^{-\tilde{p}}(c, d)]^{-1},$$

the desired relation yields. The case $pn \leq 1$ follows in the same way.

(ii) Let $p \in (-\frac{1}{n}, 0)$. Since $-\tilde{p} = -\frac{p}{pn+1} \in (0, \infty)$ and $p = \frac{\tilde{p}}{\tilde{p}n+1}$, we can apply (i) by reversing the roles of the means.

(iii) We first assume that $a \geq b$. Then we have

$$\frac{[\mathcal{M}_s^{\frac{1}{n}}(\frac{1}{c}, \frac{1}{d})]^{\frac{1}{n}}}{[\mathcal{M}_s^{+\infty}(a, b)]^{\frac{1}{n}} [\mathcal{M}_s^{\frac{1}{n}}(\frac{1}{ac}, \frac{1}{bd})]^{\frac{1}{n}}} = 1 + s \frac{(b^{\frac{1}{n}} - a^{\frac{1}{n}})}{(abd)^{\frac{1}{n}}} \left[\mathcal{M}_s^{-\frac{1}{n}}(ac, bd) \right]^{\frac{1}{n}}.$$

After a rearrangement, Bernoulli's inequality and (2.1) give the required inequality. The same can be done for $a \leq b$.

(iv) We first assume that $s \geq \frac{1}{2}$, i.e., $\min(s, 1-s) = 1-s$. We apply (2.2) with $r = \frac{1}{1-s} \geq 2$ and $r' = \frac{1}{s}$, obtaining

$$\begin{aligned} \frac{[\mathcal{M}_s^0(\frac{1}{c}, \frac{1}{d})]^{\frac{1}{n}}}{[\mathcal{M}_s^0(a, b)]^{\frac{1}{n}} [\mathcal{M}_s^{\frac{1}{n}}(\frac{1}{ac}, \frac{1}{bd})]^{\frac{1}{n}}} &= \frac{(\frac{1}{ac})^{\frac{1-s}{n}} (\frac{1}{bd})^{\frac{s}{n}}}{(1-s)(\frac{1}{ac})^{\frac{1}{n}} + s(\frac{1}{bd})^{\frac{1}{n}}} \\ &\leq \frac{(1-s)(\frac{1}{ac})^{\frac{1}{n}} + s(\frac{1}{bd})^{\frac{1}{n}} - (1-s) \left| (\frac{1}{ac})^{\frac{1-s}{n}} - ((\frac{1}{bd})^{\frac{s}{n}})^{\frac{1-s}{s}} \right|^{\frac{1}{1-s}}}{(1-s)c^{\frac{1}{n}} + sd^{\frac{1}{n}}} \\ &= 1 - (1-s) \frac{\left| (\frac{1}{ac})^{\frac{1-s}{n}} - (\frac{1}{bd})^{\frac{1-s}{n}} \right|^{\frac{1}{1-s}}}{(1-s)(\frac{1}{ac})^{\frac{1}{n}} + s(\frac{1}{bd})^{\frac{1}{n}}} \\ &= 1 - (1-s) \left| \left[\mathcal{M}_s^{-\frac{1}{n}} \left(1, \frac{bd}{ac} \right) \right]^{\frac{1-s}{n}} - \left[\mathcal{M}_s^{-\frac{1}{n}} \left(\frac{ac}{bd}, 1 \right) \right]^{\frac{1-s}{n}} \right|^{\frac{1}{1-s}}. \end{aligned}$$

Rearranging the above inequality, by using (2.1) and the Bernoulli inequality, it follows that

$$\mathcal{M}_s^0(ac, bd) \geq \mathcal{M}_s^{-\frac{1}{n}}(ac, bd) \left(1 + n(1-s) \left| \left[\mathcal{M}_s^{-\frac{1}{n}} \left(1, \frac{bd}{ac} \right) \right]^{\frac{1-s}{n}} - \left[\mathcal{M}_s^{-\frac{1}{n}} \left(\frac{ac}{bd}, 1 \right) \right]^{\frac{1-s}{n}} \right|^{\frac{1}{1-s}} \right).$$

If $s \leq \frac{1}{2}$, we proceed in a similar way as above. □

Remark 2.1. (a) For $p = -\frac{1}{n}$, we set $G_s^{-\frac{1}{n},n}(a, b, c, d) = G_s^{+\infty,n}(c, d, a, b)$.

(b) Let $p \geq -\frac{1}{n}$. We point out a homogeneity property of $G_s^{p,n}(\cdot, \cdot, \cdot, \cdot)$, i.e., for every $\lambda, \mu > 0$ and $a, b, c, d > 0$, one has

$$G_s^{p,n}(\lambda a, \lambda b, \mu c, \mu d) = G_s^{p,n}(a, b, c, d).$$

3. THE BORELL-BRASCAMP-LIEB DEFICIT: PROOF OF THEOREMS 1.1 & 1.2

Let (M, w) be a complete n -dimensional Riemannian manifold and $d : M \times M \rightarrow \mathbb{R}$ be its distance function. Let $B(x, r) = \{y \in M : d(x, y) < r\}$ be the geodesic ball with center $x \in M$ and radius $r > 0$. Fix $s \in (0, 1)$. According to Cordero-Erausquin, McCann and Schmuckenschläger [12], the volume distortion coefficient in (M, w) is defined by

$$v_s(x, y) = \lim_{r \rightarrow 0} \frac{\mathfrak{m}(Z_s(x, B(y, r)))}{\mathfrak{m}(B(y, sr))}, \quad (3.1)$$

where $\mathfrak{m}(S) = \int_S dV_w$ for every measurable set $S \subset M$.

The proof of Theorems 1.1 and 1.2 will be presented simultaneously.

Proof of Theorems 1.1 and 1.2. We recall that $\psi_s : M \rightarrow M$ is the s -interpolant optimal transport map given by $\psi_s(x) = \exp_x(-s\nabla\varphi(x))$ for a.e. $x \in \text{supp } f$ (for some $d^2/2$ -concave function $\varphi : M \rightarrow \mathbb{R}$), and its Jacobian is

$$\text{Jac}(\psi_s)(x) = \det \left[Y(s) \text{Hess} \left[\frac{d^2(\psi_s(x), \cdot)}{2} - s\varphi(\cdot) \right] \Big|_x \right], \quad (3.2)$$

where $Y(s) = d(\exp_x)_{-s\nabla\varphi(x)}$ is the Jacobian of the exponential map at $-s\nabla\varphi(x) \in T_x M$. Similarly, we have that

$$\text{Jac}(\psi)(x) = \det \left[Y(1) \text{Hess} \left[\frac{d^2(\psi(x), \cdot)}{2} - \varphi(\cdot) \right] \Big|_x \right].$$

Let $A = \text{supp } f$. The Jacobian determinant inequality on (M, w) , cf. Cordero-Erausquin, McCann and Schmuckenschläger [12, Lemma 6.1], reads as

$$\text{Jac}(\psi_s)(x) \geq \mathcal{M}_s^{\frac{1}{n}}(v_{1-s}(\psi(x), x), v_s(x, \psi(x)) \text{Jac}(\psi)(x)) \quad \text{for a.e. } x \in A. \quad (3.3)$$

We notice that the Monge-Ampère equation holds, i.e.

$$\tilde{f}(x) = \tilde{g}(\psi(x)) \text{Jac}(\psi)(x) \quad \text{for a.e. } x \in A. \quad (3.4)$$

Let

$$\tilde{h}(z) = \frac{h(z)}{\mathcal{M}_s^{\frac{p}{pn+1}}(\|f\|_1, \|g\|_1)}, \quad z \in M.$$

We first notice that

$$\psi_s(A) \subseteq \text{supp } h,$$

up to a null measure set. Indeed, if $x \in A$, then $\psi(x) \in \text{supp } g$ and by the hypothesis (1.5) and convention on \mathcal{M}_s^p , it follows that $h(\psi_s(x)) > 0$. Furthermore, we also have the injectivity of the interpolant ψ_s on A , see [12, Lemma 5.3].

Case 1: $p \in (-\frac{1}{n}, \infty) \setminus \{0\}$. A change of variable $z = \psi_s(x)$ (since ψ_s is injective), Lemma 2.1 (i)&(ii), relations (1.5), (3.3) and (3.4) give

$$\begin{aligned}
\|\tilde{h}\|_1 &= \int_M \tilde{h} = \int_{\text{supp } h} \tilde{h} \\
&\geq \int_{\psi_s(A)} \tilde{h} = \int_A \tilde{h}(\psi_s(x)) \text{Jac}(\psi_s)(x) \\
&\geq \int_A \mathcal{M}_s^p \left(\frac{f(x)}{v_{1-s}(\psi(x), x)}, \frac{g(\psi(x))}{v_s(x, \psi(x))} \right) \mathcal{M}_s^{-\frac{p}{pn+1}} \left(\frac{1}{\|f\|_1}, \frac{1}{\|g\|_1} \right) \text{Jac}(\psi_s)(x) \\
&\geq \int_A \mathcal{M}_s^{-\frac{1}{n}} \left(\frac{\tilde{f}(x)}{v_{1-s}(\psi(x), x)}, \frac{\tilde{g}(\psi(x))}{v_s(x, \psi(x))} \right) \text{Jac}(\psi_s)(x) \times \\
&\quad \times \left(1 + G_s^{p,n} \left(\frac{f(x)}{v_{1-s}(\psi(x), x)}, \frac{g(\psi(x))}{v_s(x, \psi(x))}, \frac{1}{\|f\|_1}, \frac{1}{\|g\|_1} \right) \right) \\
&\geq \int_A \mathcal{M}_s^{-\frac{1}{n}} \left(\frac{\tilde{f}(x)}{v_{1-s}(\psi(x), x)}, \frac{\tilde{g}(\psi(x))}{v_s(x, \psi(x))} \right) \mathcal{M}_s^{\frac{1}{n}} \left(v_{1-s}(\psi(x), x), v_s(x, \psi(x)) \frac{\tilde{f}(x)}{\tilde{g}(\psi(x))} \right) \times \\
&\quad \times \left(1 + G_s^{p,n} \left(\frac{f(x)}{v_{1-s}(\psi(x), x)}, \frac{g(\psi(x))}{v_s(x, \psi(x))}, \frac{1}{\|f\|_1}, \frac{1}{\|g\|_1} \right) \right) \\
&= 1 + \int_M \tilde{f}(x) G_s^{p,n} \left(\frac{f(x)}{v_{1-s}(\psi(x), x)}, \frac{g(\psi(x))}{v_s(x, \psi(x))}, \frac{1}{\|f\|_1}, \frac{1}{\|g\|_1} \right),
\end{aligned}$$

which proves the first claim.

Now, assume that (a) holds, i.e., $\delta_{M,s}^p(f, g, h) = \|\tilde{h}\|_1 - 1 = 0$. It follows directly that

$$G_s^{p,n} \left(\frac{f(x)}{v_{1-s}(\psi(x), x)}, \frac{g(\psi(x))}{v_s(x, \psi(x))}, \frac{1}{\|f\|_1}, \frac{1}{\|g\|_1} \right) = 0 \quad \text{for a.e. } x \in A,$$

and there are equalities in the above estimates. In particular,

$$\text{supp } \tilde{h} = \text{supp } h = \psi_s(A),$$

up to a null measure set of M , which gives property (i). Since $G_s^{p,n}(a, b, c, d) = 0$ if and only if $\frac{a}{b} = \left(\frac{d}{c}\right)^{\frac{1}{pn+1}}$, the latter relation is equivalent to

$$\frac{f(x)}{v_{1-s}(\psi(x), x) \|f\|_1^{\frac{1}{pn+1}}} = \frac{g(\psi(x))}{v_s(x, \psi(x)) \|g\|_1^{\frac{1}{pn+1}}} \quad \text{for a.e. } x \in A.$$

By (1.5) and the above estimate we necessarily have for a.e. $x \in A$ that

$$\begin{aligned}
h(\psi_s(x)) &= \mathcal{M}_s^p \left(\frac{f(x)}{v_{1-s}(\psi(x), x)}, \frac{g(\psi(x))}{v_s(x, \psi(x))} \right) \\
&= \frac{f(x)}{v_{1-s}(\psi(x), x) \|f\|_1^{\frac{1}{pn+1}}} \left[\mathcal{M}_s^{\frac{p}{pn+1}} (\|f\|_1, \|g\|_1) \right]^{\frac{1}{pn+1}},
\end{aligned}$$

which is (iii). Since we also have equality in the Jacobi determinant inequality (3.3), property (ii) directly follows by (iii); thus every item of (b) holds true.

The reverse implication is trivial.

Case 2: $p = +\infty$. A similar reasoning as in Case 1 and Lemma 2.1 (iii) give that

$$\|\tilde{h}\|_1 \geq 1 + \int_M \tilde{f}(x) G_s^{+\infty,n} \left(\frac{f(x)}{v_{1-s}(\psi(x), x)}, \frac{g(\psi(x))}{v_s(x, \psi(x))}, \frac{1}{\|f\|_1}, \frac{1}{\|g\|_1} \right).$$

If $\delta_{M,s}^{+\infty}(f, g, h) = 0$, the latter integrand is necessarily zero. Since $G_s^{+\infty,n}(a, b, c, d) = 0$ if and only if $a = b$, we obtain

$$\frac{f(x)}{v_{1-s}(\psi(x), x)} = \frac{g(\psi(x))}{v_s(x, \psi(x))} \quad \text{for a.e. } x \in A.$$

Furthermore, in order to have the equality case, by (1.5) and the latter relation we necessarily have for a.e. $x \in A$ that

$$h(\psi_s(x)) = \mathcal{M}_s^{+\infty} \left(\frac{f(x)}{v_{1-s}(\psi(x), x)}, \frac{g(\psi(x))}{v_s(x, \psi(x))} \right) = \frac{f(x)}{v_{1-s}(\psi(x), x)} = \frac{g(\psi(x))}{v_s(x, \psi(x))},$$

which corresponds to (iii). Clearly, one also has (i) and by the equality in (3.3) we necessarily have for a.e. $x \in A$ that

$$\text{Jac}(\psi_s)(x) = \mathcal{M}_s^{\frac{1}{n}} \left(v_{1-s}(\psi(x), x), v_s(x, \psi(x)) \frac{\tilde{f}(x)}{\tilde{g}(\psi(x))} \right) = \frac{v_{1-s}(\psi(x), x)}{\|f\|_1} \mathcal{M}_s^{\frac{1}{n}}(\|f\|_1, \|g\|_1),$$

which is precisely (ii). The converse is trivial again.

Case 3: $p = 0$. Similarly as above, by Lemma 2.1 (iv) we have

$$\begin{aligned} \|\tilde{h}\|_1 &\geq \int_{\psi_s(A)} \tilde{h} = \int_A \tilde{h}(\psi_s(x)) \text{Jac}(\psi_s)(x) \\ &\geq \int_A \mathcal{M}_s^0 \left(\frac{f(x)}{v_{1-s}(\psi(x), x)}, \frac{g(\psi(x))}{v_s(x, \psi(x))} \right) \mathcal{M}_s^0 \left(\frac{1}{\|f\|_1}, \frac{1}{\|g\|_1} \right) \text{Jac}(\psi_s)(x) \\ &\geq 1 + \int_M \tilde{f}(x) G_s^{0,n} \left(\frac{f(x)}{v_{1-s}(\psi(x), x)}, \frac{g(\psi(x))}{v_s(x, \psi(x))}, \frac{1}{\|f\|_1}, \frac{1}{\|g\|_1} \right). \end{aligned}$$

Let us assume that $\delta_{M,s}^0(f, g, h) = 0$; thus, the latter integrand is zero. Note that $G_s^{0,n}(a, b, c, d) = 0$ if and only if $ac = bd$. Therefore, we obtain

$$\frac{\tilde{f}(x)}{v_{1-s}(\psi(x), x)} = \frac{\tilde{g}(\psi(x))}{v_s(x, \psi(x))} \quad \text{for a.e. } x \in A.$$

Having equality in (1.5), from the latter relation we obtain for a.e. $x \in A$ that

$$h(\psi_s(x)) = \mathcal{M}_s^0 \left(\frac{f(x)}{v_{1-s}(\psi(x), x)}, \frac{g(\psi(x))}{v_s(x, \psi(x))} \right) = \frac{\tilde{f}(x)}{v_{1-s}(\psi(x), x)} \mathcal{M}_s^0(\|f\|_1, \|g\|_1),$$

which is (iii). Property (i) follows trivially, while (ii) comes from (iii) and the equality in (3.3), i.e.,

$$\text{Jac}(\psi_s)(x) = \mathcal{M}_s^{\frac{1}{n}} \left(v_{1-s}(\psi(x), x), v_s(x, \psi(x)) \frac{\tilde{f}(x)}{\tilde{g}(\psi(x))} \right) = v_{1-s}(\psi(x), x) \quad \text{for a.e. } x \in A.$$

Case 4: $p = -\frac{1}{n}$. The proof is similar to the case $p = +\infty$; indeed, one has

$$\|\tilde{h}\|_1 \geq 1 + \int_M \tilde{f}(x) G_s^{+\infty,n} \left(\frac{1}{\|f\|_1}, \frac{1}{\|g\|_1}, \frac{f(x)}{v_{1-s}(\psi(x), x)}, \frac{g(\psi(x))}{v_s(x, \psi(x))} \right).$$

By Remark 2.1, the claim follows. The equality case is treated in the following result. \square

Theorem 3.1. (Equality in Borell-Brascamp-Lieb inequality; $p = -\frac{1}{n}$) *Let us assume that the assumptions in Theorem 1.1 are fulfilled. Then the following assertions are equivalent:*

- (a) $\delta_{M,s}^{-\frac{1}{n}}(f, g, h) = 0$;
- (b) *the following statements simultaneously hold:*
 - (i) $\text{supp } h = \psi_s(\text{supp } f)$ up to a null measure set;
 - (ii) $h(\psi_s(x)) = \mathcal{M}_s^{-\frac{1}{n}} \left(\frac{f(x)}{v_{1-s}(\psi(x), x)}, \frac{g(\psi(x))}{v_s(x, \psi(x))} \right) = \frac{f(x)}{\text{Jac}(\psi_s)(x)}$ for a.e. $x \in \text{supp } f$;
 - (iii) $\|f\|_1 = \|g\|_1$.

Proof. If $\delta_{M,s}^{-\frac{1}{n}}(f, g, h) = 0$, it follows by Case 4 of the previous proof that $\|f\|_1 = \|g\|_1$. Clearly, (i) holds true again by Case 4. Finally, (ii) follows by direct computation. \square

We conclude this section by a weak stability result for geometric inequalities:

Proposition 3.1. *Let $n \geq 2$, $s \in (0, 1)$ and $p \geq -\frac{1}{n}$. For every nonempty compact sets $A, B \subset \mathbb{R}^n$ with $V(A) \neq 0 \neq V(B)$ we have*

$$\frac{V((1-s)A + sB)}{\mathcal{M}_s^{\frac{p}{1+pn}}(V(A), V(B))} - 1 \geq G_s^{p,n}(1, 1, V(B), V(A)). \quad (3.5)$$

Proof. Let $f = \mathbb{1}_A$, $g = \mathbb{1}_B$ and $h = \mathbb{1}_{(1-s)A + sB}$; then

$$\delta_{\mathbb{R}^n, s}^p(f, g, h) = \frac{V((1-s)A + sB)}{\mathcal{M}_s^{\frac{p}{1+pn}}(V(A), V(B))} - 1.$$

On the other hand, for a.e. $x \in A$, we have

$$G_s^{p,n} \left(\frac{f(x)}{v_{1-s}(\psi(x), x)}, \frac{g(\psi(x))}{v_s(x, \psi(x))}, \frac{1}{\|f\|_1}, \frac{1}{\|g\|_1} \right) = G_s^{p,n} \left(1, 1, \frac{1}{V(A)}, \frac{1}{V(B)} \right) = G_s^{p,n}(1, 1, V(B), V(A)).$$

It remains to apply Theorem 1.1, which concludes the proof of (3.5). \square

Remark 3.1. The quantitative log-Brunn-Minkowski inequality (see Corollary 1.1) directly follows by (3.5) for $p = 0$. Further quantitative results can be stated for $p \in [-\frac{1}{n}, \infty)$ where the right hand side of (3.5) measures the difference between the volumes $V(A)$ and $V(B)$. For $p = +\infty$, inequality (3.5) reduces precisely to the usual Brunn-Minkowski inequality (1.3) since $G_s^{+\infty, n}(1, 1, V(B), V(A)) = 0$.

4. APPLICATIONS: RIGIDITY RESULTS

In this section we prove several rigidity results, both in Riemannian and Finsler manifolds. The notations are kept from the previous sections. In addition, for every $k \in \mathbb{R}$, let $\mathbf{s}_k : [0, \infty) \rightarrow \mathbb{R}$ be defined by

$$\mathbf{s}_k(r) = \begin{cases} \frac{\sin(\sqrt{k}r)}{\sqrt{k}r} & \text{if } k > 0, \\ 1 & \text{if } k = 0, \\ \frac{\sinh(\sqrt{-k}r)}{\sqrt{-k}r} & \text{if } k < 0, \end{cases} \quad r > 0.$$

By limiting, one may choose $\mathbf{s}_k(0) = 1$.

4.1. Riemannian manifolds.

Theorem 4.1. (Curvature rigidity; Riemannian case) *Let (M, w) be a complete n -dimensional Riemannian manifold with Ricci curvature $\text{Ric}(M) \geq (n-1)k$ for some $k \in \mathbb{R}$. Let $s \in (0, 1)$, $p \geq -\frac{1}{n}$ and $f, g, h : M \rightarrow [0, \infty)$ be three non-zero, compactly supported integrable functions with $\text{supp } f = A$ and $\text{supp } g = B$, verifying*

$$h(z) \geq \mathcal{M}_s^p \left(\left(\frac{\mathbf{s}_k(d(x, y))}{\mathbf{s}_k((1-s)d(x, y))} \right)^{n-1} f(x), \left(\frac{\mathbf{s}_k(d(x, y))}{\mathbf{s}_k(sd(x, y))} \right)^{n-1} g(y) \right) \quad (4.1)$$

for all $(x, y) \in A \times B, z \in Z_s(x, y)$. Then $\delta_{M, s}^p(f, g, h) \geq 0$.

Moreover, if $\delta_{M, s}^p(f, g, h) = 0$ then for a.e. $x \in \text{supp } f = A$ one has:

- (i) the sectional curvature is equal to the constant k along the geodesic $t \mapsto \psi_t(x)$, $t \in [0, 1]$;
- (ii) if $p > -\frac{1}{n}$ and $d_x = d(x, \psi(x))$, then

$$\frac{h(\psi_s(x))}{\left[\mathcal{M}_s^{\frac{p}{pn+1}}(\|f\|_1, \|g\|_1) \right]^{\frac{1}{pn+1}}} = \left(\frac{\mathbf{s}_k(d_x)}{\mathbf{s}_k((1-s)d_x)} \right)^{n-1} \frac{f(x)}{\|f\|_1^{\frac{1}{pn+1}}} = \left(\frac{\mathbf{s}_k(d_x)}{\mathbf{s}_k(sd_x)} \right)^{n-1} \frac{g(\psi(x))}{\|g\|_1^{\frac{1}{pn+1}}}.$$

- (iii) if $p = -\frac{1}{n}$ and $d_x = d(x, \psi(x))$, then $\|f\|_1 = \|g\|_1$ and

$$h(\psi_s(x)) = \mathcal{M}_s^{-\frac{1}{n}} \left(\left(\frac{\mathbf{s}_k(d_x)}{\mathbf{s}_k((1-s)d_x)} \right)^{n-1} f(x), \left(\frac{\mathbf{s}_k(d_x)}{\mathbf{s}_k(sd_x)} \right)^{n-1} g(\psi(x)) \right).$$

Proof. Since $\text{Ric}(M) \geq (n-1)k$, Bishop's comparison principle implies that for every $x \in M$, $y \in M \setminus \text{cut}(x)$ and $s \in (0, 1)$,

$$v_s(x, y) \geq \left(\frac{\mathbf{s}_k(sd(x, y))}{\mathbf{s}_k(d(x, y))} \right)^{n-1}, \quad (4.2)$$

see e.g. Bishop and Crittenden [5], and Cordero-Erausquin, McCann and Schmuckenschläger [12, Corollary 2.2]. Here, $\text{cut}(x) \subset M$ denotes the cut locus of $x \in M$. The estimate (4.2) and assumption (4.1) imply through the monotonicity of $\mathcal{M}_s^p(\cdot, \cdot)$ the validity of (1.5). Consequently, Theorem 1.1 implies the fact that $\delta_{M,s}^p(f, g, h) \geq 0$.

Assume now that the Borell-Brascamp-Lieb deficit vanishes, i.e. $\delta_{M,s}^p(f, g, h) = 0$. By the proof of Theorem 1.2 (cf. Cases 1-4), one has for a.e. $x \in A$ that

$$h(\psi_s(x)) = \mathcal{M}_s^p \left(\frac{f(x)}{v_{1-s}(\psi(x), x)}, \frac{g(\psi(x))}{v_s(x, \psi(x))} \right). \quad (4.3)$$

On the other hand, by relations (4.1)-(4.3) and the monotonicity of $\mathcal{M}_s^p(\cdot, \cdot)$ we have for a.e. $x \in A$ that

$$\begin{aligned} h(\psi_s(x)) &\geq \mathcal{M}_s^p \left(\left(\frac{\mathbf{s}_k(d_x)}{\mathbf{s}_k((1-s)d_x)} \right)^{n-1} f(x), \left(\frac{\mathbf{s}_k(d_x)}{\mathbf{s}_k(sd_x)} \right)^{n-1} g(\psi(x)) \right) \\ &\geq \mathcal{M}_s^p \left(\frac{f(x)}{v_{1-s}(\psi(x), x)}, \frac{g(\psi(x))}{v_s(x, \psi(x))} \right) \\ &= h(\psi_s(x)). \end{aligned}$$

Consequently, we have equalities in the above estimates. Again, by the monotonicity of $\mathcal{M}_s^p(\cdot, \cdot)$ we necessarily have for a.e. $x \in A$ that

$$\left(\frac{\mathbf{s}_k((1-s)d_x)}{\mathbf{s}_k(d_x)} \right)^{n-1} = v_{1-s}(\psi(x), x) \quad \text{and} \quad \left(\frac{\mathbf{s}_k(sd_x)}{\mathbf{s}_k(d_x)} \right)^{n-1} = v_s(x, \psi(x)), \quad (4.4)$$

which proves (ii)&(iii) through Theorem 1.2 (b)(iii) and Theorem 3.1, respectively.

Note that (4.4) implies in particular that for a.e. $x \in A$,

$$v_s(x, \psi(x)) = \frac{\det[Y(s)]}{\det[Y(1)]} = \left(\frac{\mathbf{s}_k(sd_x)}{\mathbf{s}_k(d_x)} \right)^{n-1}.$$

Due to Bishop and Crittenden [5, §11.10], an analysis of the behavior of Jacobian fields shows that for a.e. $x \in A$ the sectional curvature along the geodesics $t \mapsto \psi_t(x)$, $t \in [0, 1]$ is constant, having the value k , which concludes the proof of (i). \square

Remark 4.1. Theorem 4.1 complements both Théorém 1 from Cordero-Erausquin [11] and Corollary 2.2 from Cordero-Erausquin, McCann and Schmuckenschläger [12] where the Prékopa-Leindler inequalities are considered (i.e., $p = 0$).

Proof of Theorem 1.3. Let

$$h := \mathbb{1}_{Z_s(A, B)}, \quad f := \left(\frac{\mathbf{s}_k((1-s)\Theta_{A, B})}{\mathbf{s}_k(\Theta_{A, B})} \right)^{n-1} \mathbb{1}_A \quad \text{and} \quad g := \left(\frac{\mathbf{s}_k(s\Theta_{A, B})}{\mathbf{s}_k(\Theta_{A, B})} \right)^{n-1} \mathbb{1}_B.$$

By monotonicity reasons it turns out that (4.1) holds. Therefore, due to Theorem 4.1, one has

$$\delta_{M,s}^{+\infty}(f, g, h) \geq 0,$$

which is precisely the generalized Brunn-Minkowski inequality (1.6).

In the sequel, let us assume that we have equality in (1.6), i.e.,

$$\mathbf{m}(Z_s(A, B))^{\frac{1}{n}} = \tau_{1-s}^{k, n}(\Theta_{A, B}) \mathbf{m}(A)^{\frac{1}{n}} + \tau_s^{k, n}(\Theta_{A, B}) \mathbf{m}(B)^{\frac{1}{n}}. \quad (4.5)$$

Moreover, by Theorem 4.1 (ii), we also have for a.e. $x \in A$ that

$$1 = \frac{\left(\frac{s_k(d_x)}{s_k((1-s)d_x)}\right)^{n-1}}{\left(\frac{s_k(\Theta_{A,B})}{s_k((1-s)\Theta_{A,B})}\right)^{n-1}} = \frac{\left(\frac{s_k(d_x)}{s_k(sd_x)}\right)^{n-1}}{\left(\frac{s_k(\Theta_{A,B})}{s_k(s\Theta_{A,B})}\right)^{n-1}}. \quad (4.6)$$

(i) (Positively curved case) Two cases are distinguished.

We first assume that $A \cap B \neq \emptyset$. Clearly, by (1.7) we have $\Theta_{A,B} = 0$. Therefore, due to the monotonicity of $r \mapsto \frac{s_k(r)}{s_k(sr)}$, relation (4.6) and $\Theta_{A,B} = 0$ give that $d_x = d(x, \psi(x)) = 0$ for a.e. $x \in A$. Thus, $\psi(x) = x$ for a.e. $x \in A$ which implies that $B = A$ up to a null measure set. Thus, (4.5) reduces to $\mathbf{m}(Z_s(A, B)) = \mathbf{m}(A) = \mathbf{m}(B)$. Let $S = A \cap B$. It is clear that $\mathbf{m}(S) = \mathbf{m}(A)$. By the definition of the s -intermediate set Z_s , we have that $S \subseteq Z_s(S, S) \subseteq Z_s(A, B)$. Moreover, $\mathbf{m}(Z_s(A, B) \setminus S) = \mathbf{m}(Z_s(A, B)) - \mathbf{m}(S) = 0$, i.e. $Z_s(A, B)$ is equal to $A \cap B$ up to a null measure set.

Now, we assume that $A \cap B = \emptyset$. By the monotonicity of $r \mapsto \frac{s_k(r)}{s_k(sr)}$ and (4.6) we have

$$d_x = d(x, \psi(x)) = \Theta_{A,B} = \min\{d(x, y) : x \in A, y \in B\} > 0 \text{ for a.e. } x \in A. \quad (4.7)$$

For simplicity of notation, let $t_0 := \Theta_{A,B}$ and

$$B_{t_0} = \{x \in M : \text{there exists } y \in B \text{ such that } d(x, y) < t_0\} = \bigcup_{y \in B} B(y, t_0)$$

be the t_0 -neighborhood of B .

It is clear that $A \cap B_{t_0} = \emptyset$. Indeed, if we assume that $x \in A \cap B_{t_0}$, then there exists $y \in B$ such that $d(x, y) < t_0$, which contradicts the fact that $t_0 = \Theta_{A,B}$.

Now, let us fix $x \in A$ such that $d(x, \psi(x)) = t_0$; due to (4.7), the latter happens for a.e. $x \in A$. By construction, we have that $B(\psi(x), t_0) \subset \text{int} B_{t_0} = B_{t_0}$, thus $B(\psi(x), t_0) \cap A = \emptyset$. Fix $r_0 \in (0, t_0)$. Then, for every $0 < r < r_0$ let us fix $z_r \in Z_{\frac{r}{2t_0}}(x, \psi(x))$; then $B(z_r, \frac{r}{2}) \subset B(x, r) \cap B(\psi(x), t_0)$. Therefore, $B(z_r, \frac{r}{2}) \subset B(x, r) \setminus A$, i.e., A is $\frac{1}{2}$ -porous at x , see Rajala [28]. In particular, the set A has null measure, $\mathbf{m}(A) = 0$, which contradicts our assumption.

(ii) (Negatively curved case) Due to (1.7), one has $\Theta_{A,B} = \max\{d(x, y) : x \in A, y \in B\} > 0$. Similarly as above, relation (4.6) implies that

$$d_x = d(x, \psi(x)) = \Theta_{A,B} =: t^0 \text{ for a.e. } x \in A. \quad (4.8)$$

The proof is 'dual' to (i); for completeness, we provide it. Let

$$B^{t^0} = \bigcup_{y \in B} (M \setminus \overline{B}(y, t^0)),$$

where $\overline{B}(y, r) = \{x \in M : d(x, y) \leq r\}$, $r > 0$. Since $t^0 > \inf_{x \notin B} \max_{y \in B} d(x, y)$, it turns out that $\bigcap_{y \in B} \overline{B}(y, t^0) \neq \emptyset$; thus B^{t^0} is a proper open subset of M .

We claim that $A \cap B^{t^0} = \emptyset$; indeed, if $x \in A \cap B^{t^0}$, it follows that there exists $y \in B$ such that $x \in M \setminus \overline{B}(y, t^0)$, i.e., $d(x, y) > t^0$, which contradicts the definition of $t^0 = \Theta_{A,B}$.

According to (4.8), for a.e. $x \in A$, one has $d(x, \psi(x)) = t^0$ and $x \notin \text{cut}(\psi(x))$; let us choose such an $x \in A$. It is clear that $M \setminus \overline{B}(\psi(x), t^0) \subset \text{int} B^{t^0} = B^{t^0}$, thus $(M \setminus \overline{B}(\psi(x), t^0)) \cap A = \emptyset$. Since $x \notin \text{cut}(\psi(x))$, we may extend the minimal geodesic joining the point $\psi(x)$ to x beyond x such that the extended geodesic is still minimizing between $\psi(x)$ and points in a small neighborhood of x . Let $z_r \in M$ be such a point belonging to the extended geodesic with $d(z_r, x) = \frac{r}{2}$ for sufficiently small $r > 0$; thus, $d(z_r, \psi(x)) = d(z_r, x) + d(x, \psi(x)) = \frac{r}{2} + t^0$. This construction shows that $B(z_r, \frac{r}{2}) \subset B(x, r)$ and $B(z_r, \frac{r}{2}) \subset M \setminus \overline{B}(\psi(x), t^0)$, i.e., $B(z_r, \frac{r}{2}) \subset B(x, r) \setminus A$, which means that A is $\frac{1}{2}$ -porous at x . Consequently, one has $\mathbf{m}(A) = 0$, which contradicts our assumption.

(iii) (Null curved case) Since $k = 0$, by Theorem 4.1(i), for a.e. $x \in A$ the sectional curvature is zero along the geodesic $t \mapsto \psi_t(x)$, $t \in [0, 1]$. \square

Some remarks are in order after the proof of Theorem 1.3.

Remark 4.2. (a) Let $\mu = \mathbb{1}_A/\mathfrak{m}(A)dV_w$ and $\nu = \mathbb{1}_B/\mathfrak{m}(B)dV_w$ be the measures from the proof of Theorem 1.3 and $\psi : M \rightarrow M$ be the optimal transport map between them. Then we generically have the two-sided estimate for the Wasserstein distance between μ and ν ; namely,

$$(\Theta_{A,B}^{\min})^2 \leq \mathcal{W}(\mu, \nu) := \int_A d^2(x, \psi(x))d\mu(x) \leq (\Theta_{A,B}^{\max})^2, \quad (4.9)$$

where

$$\Theta_{A,B}^{\min} = \min\{d(x, y) : x \in A, y \in B\} \quad \text{and} \quad \Theta_{A,B}^{\max} = \max\{d(x, y) : x \in A, y \in B\}.$$

The proof of (i) in Theorem 1.3 treats actually the equality case at the left hand side of (4.9). Roughly speaking, when A and B are two disjoint positive measure sets, such an equality does not hold since the target measure ν cannot be reached by push-forwarding the measure μ ; the transport cost $(\Theta_{A,B}^{\min})^2$ is not enough to realize this transportation.

A similar explanation works also in the 'dual' case (ii); here, the equality in the distorted Brunn-Minkowski inequality corresponds to the equality at the right hand side of (4.9). In this setting, such an equality cannot be realized since by push-forwarding the measure μ to ν the transport cost $(\Theta_{A,B}^{\max})^2$ is too large; in fact, either we transport (a positive mass of) μ beyond ν , or we use non-optimal paths to reach ν from μ .

(b) With f, g and h from the proof of Theorem 1.3, we also have for every $p \geq -\frac{1}{n}$ that

$$\mathfrak{m}(Z_s(A, B)) \geq \mathcal{M}_s^{\frac{p}{1+np}} \left(\left(\frac{\mathbf{s}_k((1-s)\Theta_{A,B})}{\mathbf{s}_k(\Theta_{A,B})} \right)^{n-1} \mathfrak{m}(A), \left(\frac{\mathbf{s}_k(s\Theta_{A,B})}{\mathbf{s}_k(\Theta_{A,B})} \right)^{n-1} \mathfrak{m}(B) \right).$$

For $p = +\infty$ the latter inequality reduces to the distorted Brunn-Minkowski inequality (1.6).

Corollary 4.1. (Equality in Brunn-Minkowski inequality vs flatness) *Let (M, w) be a complete n -dimensional Riemannian manifold with non-negative Ricci curvature and $s \in (0, 1)$. Then for every nonempty open bounded sets $A, B \subset M$ one has*

$$\mathfrak{m}(Z_s(A, B))^{\frac{1}{n}} \geq (1-s)\mathfrak{m}(A)^{\frac{1}{n}} + s\mathfrak{m}(B)^{\frac{1}{n}}. \quad (4.10)$$

Furthermore, we have:

- (i) if equality holds in (4.10) for arbitrary (small) geodesic balls $A = B(x, r)$ and $B = B(y, R)$ with $x, y \in M$ and $r, R > 0$, then (M, w) is flat;
- (ii) if (M, g) is simply connected, equality holds in (4.10) for arbitrary geodesic balls $A = B(x, r)$ and $B = B(y, R)$ if and only if (M, w) is isometric to \mathbb{R}^n .

Proof. (i) Assume that we have equality in (4.10) for every geodesic balls $A = B(x, r)$ and $B = B(y, R)$ with $x, y \in M$ and $r, R > 0$ sufficiently small. Let $\psi : A \rightarrow B$ be the optimal transport map from the measure $\mu = \mathbb{1}_A/\mathfrak{m}(A)dV_w$ to $\nu = \mathbb{1}_B/\mathfrak{m}(B)dV_w$. By Theorem 1.3(iii), the sectional curvature is zero along the geodesics $t \mapsto \psi_t(x)$, $t \in [0, 1]$, joining a.e. $x \in A$ to $\psi(x) \in B$. The arbitrariness of the sets A and B and a density argument shows that the sectional curvature on (M, w) is zero.

(ii) If (M, w) is isometric to \mathbb{R}^n , we have equality in (4.10) for every ball. Conversely, if (M, w) is simply connected, the equality case in (4.10) for geodesic balls implies that (M, w) has zero sectional curvature (from (i)). By the Killing-Hopf theorem it follows that (M, w) is isometric to \mathbb{R}^n . \square

We conclude this subsection by presenting a new proof for the characterization of equality in Brunn-Minkowski-type inequalities in \mathbb{R}^n . Here, the main point is the use of the optimal mass transportation theory and a regularity result of Caffarelli [8].

Corollary 4.2. (Equality in Brunn-Minkowski-type inequalities in \mathbb{R}^n) *Let $s \in (0, 1)$, $p \geq -\frac{1}{n}$, and $A, B \subset \mathbb{R}^n$ be two convex bodies. Then,*

$$V((1-s)A + sB) \geq \mathcal{M}_s^{\frac{p}{1+np}}(V(A), V(B)). \quad (4.11)$$

Moreover,

- (i) if $p = +\infty$, equality holds in (4.11) if and only if the sets A and B are homothetic;
- (ii) if $p < +\infty$, equality holds in (4.11) if and only if the sets A and B are translates.

Proof. Inequality (4.11) directly follows by Theorem 1.1, choosing the indicator functions $f = \mathbb{1}_A$, $g = \mathbb{1}_B$ and $h = \mathbb{1}_{(1-s)A+sB}$ of the sets A , B and $Z_s(A, B) = (1-s)A + sB$, respectively. Let $\psi : A \rightarrow B$ be the optimal transport map from the measure $\mu = \mathbb{1}_A/V(A)d\mathcal{L}^n$ to $\nu = \mathbb{1}_B/V(B)d\mathcal{L}^n$; in fact, $\psi(x) = \exp_x(-\nabla\varphi(x)) = x - \nabla\varphi(x)$ for some $\|\cdot\|^2/2$ -concave function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$. Equivalently, there exists a convex function $\eta : \mathbb{R}^n \rightarrow \mathbb{R}$, $\eta(x) = \frac{\|x\|^2}{2} - \varphi(x)$ such that $\psi = \nabla\eta$ and $\psi_{\#}\mu = \nu$, see Villani [33, p.187]. Accordingly, $\psi_s(x) = x - s\nabla\varphi(x) = (1-s)x + s\nabla\eta(x)$.

Now, we assume the equality in (4.11).

First, let $p \in (-\frac{1}{n}, +\infty]$. With the above notations, Theorem 1.2 implies:

$$(1-s)A + sB = \{(1-s)x + s\nabla\eta(x) : \text{a.e. } x \in A\} \text{ up to a null } \mathcal{L}^n - \text{measure set}, \quad (4.12)$$

$$\det[(1-s)I_n + sD_o^2\eta(x)] = \left[\mathcal{M}_s^{\frac{p}{pn+1}} \left(1, \frac{V(B)}{V(A)} \right) \right]^{\frac{pn}{pn+1}} \text{ for a.e. } x \in A, \quad (4.13)$$

and

$$\frac{1}{\left[\mathcal{M}_s^{\frac{p}{pn+1}}(V(A), V(B)) \right]^{\frac{1}{pn+1}}} = \frac{1}{V(A)^{\frac{1}{pn+1}}} = \frac{1}{V(B)^{\frac{1}{pn+1}}}. \quad (4.14)$$

Here, I_n is the $n \times n$ unit matrix and D_o^2 the Aleksandrov second derivative, see e.g. Villani [33].

Since B is convex, the regularity result of Caffarelli [8] (see also Villani [33, Theorem 4.14]) implies that η is of class C^2 , thus the Aleksandrov second derivative $D_o^2\eta$ becomes the usual Hessian of η .

(i) Let $p = +\infty$. By relation (4.13) and the Monge-Ampère equation $\det[\text{Hess}\eta(x)] = \frac{V(B)}{V(A)}$ for $x \in A$, we have that

$$\det^{\frac{1}{n}}[(1-s)I_n + s\text{Hess}\eta(x)] = 1 - s + s \frac{V(B)^{\frac{1}{n}}}{V(A)^{\frac{1}{n}}} = (1-s)\det^{\frac{1}{n}}[I_n] + s\det^{\frac{1}{n}}[\text{Hess}\eta(x)], \quad x \in A.$$

The latter relation and the concavity of $\det^{\frac{1}{n}}(\cdot)$ over the cone of non-negative definite symmetric matrices (based on the arithmetic-geometric inequality) give that $\text{Hess}\eta(x) = c_0 I_n$ for every $x \in A$, where $c_0 = \left(\frac{V(B)}{V(A)} \right)^{1/n}$. By the above regularity, we have in fact that for some $x_0 \in \mathbb{R}^n$,

$$\nabla\eta(x) = c_0 x + x_0, \quad x \in A.$$

In particular, $B = \nabla\eta(A) = c_0 A + x_0$, i.e., the sets A and B are homothetic. The converse statement is trivial.

(ii) Let $p \in (-\frac{1}{n}, +\infty)$. Note that the computations in (i) are still valid, thus $B = c_0 A + x_0$ for some $x_0 \in \mathbb{R}^n$ with $c_0 = \left(\frac{V(B)}{V(A)} \right)^{1/n}$. Now, since $p < +\infty$, by (4.14) we have $V(A) = V(B)$, thus $c_0 = 1$. Consequently, $B = A + x_0$.

It remains to discuss the case $p = -\frac{1}{n}$. By Theorem 3.1, we have

$$(1-s)A + sB = \{(1-s)x + s\nabla\eta(x) : \text{a.e. } x \in A\} \text{ up to a null } \mathcal{L}^n - \text{measure set}, \quad (4.15)$$

$$\det[(1-s)I_n + sD_o^2\eta(x)] = 1 \text{ for a.e. } x \in A, \quad (4.16)$$

and

$$V(A) = V(B). \quad (4.17)$$

Repeating the above argument, we obtain that $B = A + x_0$ for some $x_0 \in \mathbb{R}^n$. \square

Remark 4.3. According to Corollary 4.2, the equality in the Borell-Brascamp-Lieb inequality in \mathbb{R}^n (applied for indicator functions of the sets A and B) implies relations (4.12) and (4.15), i.e.,

$$(1-s)A + sB = \{(1-s)x + s\nabla\eta(x) : x \in A\}.$$

The latter relation is nothing but the well known result of Alesker, Dar and Milman [1] concerning the parametrization of the Minkowski sum of the convex bounded sets A and B in \mathbb{R}^n by means of a suitable diffeomorphism $\psi = \nabla\eta : A \rightarrow B$; see also Villani [33, Theorem 6.9].

4.2. Finsler manifolds. In this subsection we shortly discuss some Finslerian counterparts of the results from the previous sections. To do so, we first recall those notions that are required to present our results, see Bao, Chern and Shen [4], Kristály [20], Ohta [26] and Shen [30] for details.

Let M be a connected n -dimensional smooth manifold and $TM = \bigcup_{x \in M} T_x M$ be its tangent bundle. The pair (M, F) is a *Finsler manifold* if the continuous function $F : TM \rightarrow [0, \infty)$ satisfies the conditions

- (a) $F \in C^\infty(TM \setminus \{0\})$;
- (b) $F(x, ty) = tF(x, y)$ for all $t \geq 0$ and $(x, y) \in TM$;
- (c) $g_{ij}(x, y) := [\frac{1}{2}F^2]_{y^i y^j}(x, y)$ is positive definite for all $(x, y) \in TM \setminus \{0\}$.

If $F(x, ty) = |t|F(x, y)$ for all $t \in \mathbb{R}$ and $(x, y) \in TM$, then (M, F) is a reversible Finsler manifold. A Finsler manifold (M, F) is a:

- *Riemannian manifold*, whenever $g_{ij}(x) = g_{ij}(x, y)$ is independent of y .
- *locally Minkowski space*, if there exists a local coordinate system (x^i) on M with induced tangent space coordinates (y^i) such that F depends only on $y = y^i \partial / \partial x^i$ and not on x .
- *Minkowski space*, whenever M is a finite dimensional vector space (identified by \mathbb{R}^n) which is endowed by a Minkowski norm, inducing a Finsler metric on \mathbb{R}^n by translations.
- *Berwald space*, whenever the coefficients $\Gamma_{ij}^k(x, y)$ of the Chern connection in natural coordinates are independent of y . It is clear that Riemannian manifolds and (locally) Minkowski spaces are Berwald spaces.

The Chern connection (replacing the well known Levi-Civita connection from Riemannian geometry) is a torsion free and almost metric-compatible linear connection on the pull-back bundle $\pi^* TM$ of the tangent bundle TM generated by the natural projection $\pi : TM \setminus \{0\} \rightarrow M$, see Bao, Chern and Shen [4, p. 28]. The vectors of the pull-back bundle $\pi^* TM$ are denoted by $(v; u)$ with $(x, y) = v \in TM \setminus \{0\}$ and $u \in T_x M$. For simplicity, let $\partial_i|_v = (v; \partial / \partial x^i|_x)$ be the natural local basis for $\pi^* TM$, where $v \in T_x M$. One can introduce on $\pi^* TM$ the fundamental tensor g by

$$g^v := g(\partial_i|_v, \partial_j|_v) = g_{ij}(x, y), \quad (4.18)$$

where $v = y^i (\partial / \partial x^i)|_x$.

The Chern connection induces in a natural way on $\pi^* TM$ the curvature tensor R , Jacobi fields, geodesics and parallel transport on (M, F) , see Bao, Chern and Shen [4, Chapter 3]. Geodesics are considered to be parametrized proportionally to arc-length. The Finsler manifold (M, F) is forward (resp. backward) geodesically complete if every geodesic segment can be extended to $[0, \infty)$ (resp. to $(-\infty, 0]$).

Let $u, v \in T_x M$ be two non-collinear vectors and $\mathcal{S} = \text{span}\{u, v\} \subset T_x M$. The flag curvature of the flag $\{\mathcal{S}, v\}$ is defined by

$$K(\mathcal{S}; v) = \frac{g(R(U, V)V, U)}{g(V, V)g(U, U) - g(U, V)^2}, \quad (4.19)$$

where $U = (v; u), V = (v; v) \in \pi^* TM$ and $g = g^v$, see (4.18). If (M, F) is Riemannian, the flag curvature notion reduces to the usual sectional curvature.

Let $v \in T_x M$ be a unit vector (i.e., $F(x, v) = 1$) and $\{e_i\}_{i=1, \dots, n}$ with $e_n = v$ be a basis for $T_x M$ such that $\{(v; e_i)\}_{i=1, \dots, n}$ is an orthonormal basis for $\pi^* TM$. Let $\mathcal{S}_i = \text{span}\{e_i, v\}$, $i = 1, \dots, n-1$. The Ricci curvature $\text{Ric}_F : TM \rightarrow \mathbb{R}$ is defined by $\text{Ric}_F(v) = \sum_{i=1}^{n-1} K(\mathcal{S}_i; v)$.

Let $d_F : M \times M \rightarrow \mathbb{R}$ be the natural distance function on (M, F) , and $B^+(x_0, r) = \{x \in M : d_F(x_0, x) < r\}$ and $B^-(x_0, r) = \{x \in M : d_F(x, x_0) < r\}$ the forward and backward geodesic balls with center $x_0 \in M$ and radius $r > 0$, respectively. The set of s -intermediate points Z_s on (M, F) is defined w.r.t. the metric d_F .

Let us fix the normalized volume form dV_F on (M, F) , defined by

$$dV_F(x) = \frac{\omega_n}{V(B_x(1))} dx^1 \wedge \dots \wedge dx^n, \quad (4.20)$$

where $B_x(1) = \{y = (y^i) : F(x, y^i \partial / \partial x^i) < 1\}$ is the unit tangent ball at $T_x M$, ω_n is the volume of the unit ball in \mathbb{R}^n , and V denotes the usual Euclidean volume. Note that when (M, F) is reversible then

dV_F agrees with the usual Busemann-Hausdorff measure. Let $\mathbf{m}_F(S) = \int_S dV_F(x)$ be the Finslerian volume of the open set $S \subset M$. When (\mathbb{R}^n, F) is a Minkowski space, then on account of (4.20), $dV_F = d\mathcal{L}^n$, where \mathcal{L}^n is the usual Lebesgue measure in \mathbb{R}^n .

Let $\{e_i\}_{i=1,\dots,n}$ be a basis for $T_x M$ and $g_{ij}^v = g^v(e_i, e_j)$. The mean distortion $\mu : TM \setminus \{0\} \rightarrow (0, \infty)$ is defined by $\mu(v) = \frac{V(B_x(1))}{\omega_n} \sqrt{\det(g_{ij}^v)}$. The mean covariation $H : TM \setminus \{0\} \rightarrow \mathbb{R}$ is defined by $H(v) = \frac{d}{dt}(\ln \mu(\dot{\sigma}_v(t)))|_{t=0}$, where σ_v is the geodesic such that $\sigma_v(0) = x$ and initial vector $\dot{\sigma}_v(0) = v$. We say that (M, F) is with vanishing mean covariation if H is identically zero.

The Legendre transform $J^* : T^*M \rightarrow TM$ associates to each element $\alpha \in T_x^*M$ the unique maximizer on $T_x M$ of the map $y \mapsto \alpha(y) - \frac{1}{2}F^2(x, y)$. Let $u : M \rightarrow \mathbb{R}$ be a differentiable function in the distributional sense. The Finslerian gradient of u is defined by $\nabla u(x) = J^*(x, Du(x))$, where $Du(x) \in T_x^*M$ denotes the (distributional) derivative of u at $x \in M$. In general, $u \mapsto \nabla u$ is not linear.

Given μ and ν two absolutely continuous measures on (M, F) w.r.t. dV_F with compact support, there exists a unique optimal transport map from μ to ν of the form $\psi(x) = \exp_x(\nabla(-\varphi(x)))$, where $\varphi : M \rightarrow \mathbb{R}$ is a $d_F^2/2$ -concave function on M , see Ohta [26, Theorem 4.10]. For $s \in (0, 1)$ fixed, let $\psi_s(x) = \exp_x(s\nabla(-\varphi(x)))$ be the s -intermediate optimal transport map. The key tool to prove Borell-Brescamp-Lieb inequalities on Finsler manifolds is the Jacobian inequality

$$\text{Jac}(\psi_s)(x) \geq \mathcal{M}_s^{\frac{1}{n}}(v_s^>(x, \psi(x)), v_s^<(x, \psi(x)) \text{Jac}(\psi)(x)) \quad \text{for a.e. } x \in \text{supp}(\mu), \quad (4.21)$$

where $\text{Jac}(\psi_s)(x)$ and $\text{Jac}(\psi)(x)$ are the Jacobian determinant of ψ_s and ψ at x and

$$v_s^>(x, y) = \lim_{r \rightarrow 0} \frac{\mathbf{m}_F(Z_s(B^-(x, r), y))}{\mathbf{m}_F(B^-(x, (1-s)r))} \quad \text{and} \quad v_s^<(x, y) = \lim_{r \rightarrow 0} \frac{\mathbf{m}_F(Z_s(x, B^+(y, r)))}{\mathbf{m}_F(B^+(x, sr))},$$

see Ohta [26, Proposition 5.3].

Following the arguments from [26], one can formulate in a natural way the Finslerian counterparts of Theorems 1.1 & 1.2, respectively; we leave them for the interested reader. In the sequel, we shall explicitly state a Finslerian version of Theorem 4.1, which reads as follows:

Theorem 4.2. (Curvature rigidity; Finsler case) *Let (M, F) be a forward geodesically complete, n -dimensional Finsler manifold with vanishing mean covariation and Ricci curvature $\text{Ric}_F(v) \geq (n-1)k$ for every unit vector $v \in TM$ and some $k \in \mathbb{R}$. Let $s \in (0, 1)$, $p \geq -\frac{1}{n}$ and $f, g, h : M \rightarrow [0, \infty)$ be three non-zero, compactly supported integrable functions with $\text{supp } f = A$ and $\text{supp } g = B$, verifying*

$$h(z) \geq \mathcal{M}_s^p \left(\left(\frac{\mathbf{s}_k(d_F(x, y))}{\mathbf{s}_k((1-s)d_F(x, y))} \right)^{n-1} f(x), \left(\frac{\mathbf{s}_k(d_F(x, y))}{\mathbf{s}_k(sd_F(x, y))} \right)^{n-1} g(y) \right) \quad (4.22)$$

for all $(x, y) \in A \times B, z \in Z_s(x, y)$. Then $\delta_{M,s}^p(f, g, h) \geq 0$.

Moreover, if $\delta_{M,s}^p(f, g, h) = 0$ then for a.e. $x \in \text{supp } f = A$, one has

- (i) the flag curvature is equal to the constant k along the geodesic $t \mapsto \psi_t(x)$, $t \in [0, 1]$, for flags having the form $\{\mathcal{S}, v\}$ with $\mathcal{S} = \text{span}\{u, v\} \subset T_{\psi_t(x)}M$ and $v = \frac{d}{dt}\psi_t(x)$;
- (ii) if $p > -\frac{1}{n}$ and $d_x = d_F(x, \psi(x))$, then

$$\frac{h(\psi_s(x))}{\left[\mathcal{M}_s^{\frac{p}{pn+1}}(\|f\|_1, \|g\|_1) \right]^{\frac{1}{pn+1}}} = \left(\frac{\mathbf{s}_k(d_x)}{\mathbf{s}_k((1-s)d_x)} \right)^{n-1} \frac{f(x)}{\|f\|_1^{\frac{1}{pn+1}}} = \left(\frac{\mathbf{s}_k(d_x)}{\mathbf{s}_k(sd_x)} \right)^{n-1} \frac{g(\psi(x))}{\|g\|_1^{\frac{1}{pn+1}}}.$$

- (iii) if $p = -\frac{1}{n}$ and $d_x = d_F(x, \psi(x))$, then $\|f\|_1 = \|g\|_1$ and

$$h(\psi_s(x)) = \mathcal{M}_s^{-\frac{1}{n}} \left(\left(\frac{\mathbf{s}_k(d_x)}{\mathbf{s}_k((1-s)d_x)} \right)^{n-1} f(x), \left(\frac{\mathbf{s}_k(d_x)}{\mathbf{s}_k(sd_x)} \right)^{n-1} g(\psi(x)) \right).$$

Proof. By the assumptions on the vanishing mean covariation and Ricci curvature on (M, F) , one has

$$v_s^>(x, y) \geq \left(\frac{\mathbf{s}_k((1-s)d_F(x, y))}{\mathbf{s}_k(d_F(x, y))} \right)^{n-1} \quad \text{and} \quad v_s^<(x, y) \geq \left(\frac{\mathbf{s}_k(sd_F(x, y))}{\mathbf{s}_k(d_F(x, y))} \right)^{n-1},$$

for every $s \in (0, 1)$, $x \in M$ and $y \in M \setminus \text{cut}_F(x)$, where $\text{cut}_F(x)$ is the cut locus of x in (M, F) , see Ohta [26, Theorem 7.3] and Shen [30, Theorem 1.1]. The conclusion $\delta_{M,s}^p(f, g, h) \geq 0$ follows as above.

Now, assume we have $\delta_{M,s}^p(f, g, h) = 0$. The proof of items (ii) & (iii) are similar to those from Theorem 4.1. In particular, it follows that for a.e. $x \in A$ we have

$$v_s^>(x, \psi(x)) = \left(\frac{\mathbf{s}_k((1-s)d_x)}{\mathbf{s}_k(d_x)} \right)^{n-1} \quad \text{and} \quad v_s^<(x, \psi(x)) = \left(\frac{\mathbf{s}_k(sd_x)}{\mathbf{s}_k(d_x)} \right)^{n-1}, \quad (4.23)$$

where $d_x = d_F(x, \psi(x))$. Fix $x \in A$ that verifies relations (4.23). By the equality case in [26, 30], the latter equalities imply that any Jacobi field along the geodesic $t \mapsto \psi_t(x)$, $t \in [0, 1]$, has the form $J(t) = \mathbf{s}_k(d_x t)W(d_x t)$, where W is a parallel vector field along $t \mapsto \psi_t(x)$. By the Jacobi equation we have at once that the curvature tensor along $t \mapsto \psi_t(x)$, $t \in [0, 1]$ has the property

$$R(J(t), V(t))V(t) = k d_x^2 \mathbf{s}_k(d_x t)W(d_x t), \quad t \in [0, 1],$$

where $V(t) = \frac{d}{dt}\psi_t(x)$. In particular, since $g(V(t), V(t)) = F^2(V(t)) = d_x^2$, the flag curvature for the flag $\mathcal{S} = \text{span}\{J(t), V(t)\}$ is $K(\mathcal{S}; V(t)) = k$, see (4.19), which ends the proof of (i). \square

Remark 4.4. In Theorem 4.2 (i) we have information only on the flag curvature in specific directions of the flag. When (M, F) is Riemannian, Theorem 4.2 reduces to Theorem 4.1.

Corollary 4.3. (Brunn-Minkowski inequality on Berwald spaces) *Let (M, F) be a forward geodesically complete n -dimensional Berwald space with non-negative Ricci curvature and $s \in (0, 1)$. Then for every nonempty open bounded sets $A, B \subset M$ one has*

$$\mathbf{m}_F(Z_s(A, B))^{\frac{1}{n}} \geq (1-s)\mathbf{m}_F(A)^{\frac{1}{n}} + s\mathbf{m}_F(B)^{\frac{1}{n}}. \quad (4.24)$$

If the equality holds in (4.24) for arbitrary forward geodesic balls A and B , then (M, F) is a locally Minkowski space.

Proof. Since the Ricci curvature is non-negative, we have that $v_s^< \geq 1$ and $v_s^> \geq 1$. Moreover, since every Berwald space has vanishing mean covariation, see Shen [30, Propositions 2.6 & 2.7], we may apply Theorem 4.2. Thus, (4.24) follows by the first part of Theorem 4.2 by choosing $p = +\infty$ and the indicator functions $f = \mathbb{1}_A$, $g = \mathbb{1}_B$ and $h = \mathbb{1}_{Z_s(A, B)}$ of the sets A , B and $Z_s(A, B)$, respectively.

If we have equality in (4.24) for every forward geodesic balls A and B , it turns out by Theorem 4.2(i) that the flag curvature is identically zero (being zero for every choice of the flag). Since (M, F) is a Berwald space, the vanishing of the flag curvature implies that (M, F) is locally Minkowski, see Bao, Chern and Shen [4, Section 10.5]. \square

Example 4.1. On \mathbb{R}^{n-1} ($n \geq 2$) we introduce a complete Riemannian metric w such that (\mathbb{R}^{n-1}, w) has non-negative Ricci curvature, and for every $\varepsilon \geq 0$, we define on $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ the metric $F_\varepsilon : T\mathbb{R}^n = \mathbb{R}^{2n} \rightarrow [0, \infty)$ for every $(x, t) \in \mathbb{R}^n$ and $(y, v) \in T_x\mathbb{R}^{n-1} \times T_t\mathbb{R} = \mathbb{R}^n$ by

$$F_\varepsilon((x, t), (y, v)) = \sqrt{w_x(y, y) + v^2 + \varepsilon \sqrt{w_x(y, y)^2 + v^4}}.$$

$(\mathbb{R}^n, F_\varepsilon)$ is a Riemannian manifold if and only if $\varepsilon = 0$; however, if $\varepsilon > 0$, then $(\mathbb{R}^n, F_\varepsilon)$ is a non-compact, complete, reversible non-Riemannian Berwald space with non-negative Ricci curvature.

Fix $\varepsilon > 0$. According to Corollary 4.3, if equality holds in (4.24) for arbitrary geodesic balls A and B in $(\mathbb{R}^n, F_\varepsilon)$, then $(\mathbb{R}^n, F_\varepsilon)$ is a (locally) Minkowski space, i.e., w_x is independent of x .

Minkowski spaces are the simplest non-Euclidean Finsler structures. However, it turns out that the equality in the Brunn-Minkowski inequality on generic Minkowski spaces is not automatically verified even for forward and backward geodesic balls. In addition, in Example 4.2 we provide two classes of Minkowski spaces where equalities in the Brunn-Minkowski inequality generate two genuinely different scenarios.

Corollary 4.4. (Brunn-Minkowski inequality on Minkowski spaces) *Let (\mathbb{R}^n, F) be a Minkowski space, $s \in (0, 1)$ and $A, B \subset M$ nonempty open bounded sets. Then the inequality (4.24) holds; moreover, if A and B are convex sets (in the usual sense), equality holds in (4.24) if and only if A and B are homothetic. If $x, y \in \mathbb{R}^n$ and $r, R > 0$ are fixed, the following statements are equivalent:*

- (i) *equality holds in (4.24) for $A = B^+(x, r)$ and $B = B^-(y, R)$;*
- (ii) *$B^-(y - x_0, R) = B^+(\frac{R}{r}x, R)$ for some $x_0 \in \mathbb{R}$.*

Proof. Any Minkowski space is a forward/backward complete Berwald space with zero flag curvature; thus Corollary 4.3 applies, yielding the validity of (4.24).

Assume that for the convex sets A and B we have equality in (4.24). Note that the geodesics in (\mathbb{R}^n, F) are straight lines and for every $x, y \in \mathbb{R}^n$ one has $d_F(x, y) = F(y - x)$. Thus, the positive homogeneity of F implies that $Z_s(A, B) = (1 - s)A + sB$. We also recall that $\mathfrak{m}_F(S) = \mathcal{L}^n(S)$ for every measurable set $S \subset M$. Accordingly, the equality in (4.24) can be transposed to an equality in the Euclidean Brunn-Minkowski inequality for A and B , obtaining by Corollary 4.2 that A and B are homothetic.

In the sequel, let $A = B^+(x, r)$ and $B = B^-(y, R)$ for some $x, y \in \mathbb{R}^n$ and $r, R > 0$.

(i) \Rightarrow (ii). Assume we have equality in (4.24) for A and B . Note that these sets are strictly convex domains of \mathbb{R}^n in the usual sense, both of them inheriting the convexity of the Minkowski norm F , see e.g. Bao, Chern and Shen [4, p. 12]. Accordingly, from the first part of the proof, the sets A and B are homothetic, i.e., $B^-(y, R) = c_0 B^+(x, r) + x_0$, for some $c_0 > 0$ and $x_0 \in \mathbb{R}^n$. Moreover, it follows that $c_0 = \frac{R}{r}$, thus $B^-(y - x_0, R) = B^+(\frac{R}{r}x, R)$.

(ii) \Rightarrow (i). Since $B^-(y - x_0, R) = B^+(\frac{R}{r}x, R)$, we have that $Z_s(A, B) = (1 - s)B^+(x, r) + sB^-(y, R) = sx_0 + ((1 - s)\frac{r}{R} + s)B^+(\frac{R}{r}x, R)$. Thus,

$$\mathfrak{m}_F(Z_s(A, B))^{\frac{1}{n}} = \left((1 - s)\frac{r}{R} + s \right) R \omega_n^{\frac{1}{n}} = (1 - s)r \omega_n^{\frac{1}{n}} + sR \omega_n^{\frac{1}{n}} = (1 - s)\mathfrak{m}_F(A)^{\frac{1}{n}} + s\mathfrak{m}_F(B)^{\frac{1}{n}},$$

which concludes the proof. \square

For simplicity, in the following example we restrict our argument to two-dimensional objects.

Example 4.2. (a) (*Randers-type Minkowski plane*) Let $F_b : T\mathbb{R}^2 \rightarrow [0, \infty)$ be defined by

$$F_b(x, y) := F_b(y) = \sqrt{\langle Ay, y \rangle} + \langle b, y \rangle, \quad (x, y) \in T\mathbb{R}^2, \quad (4.25)$$

where A is a 2×2 positive definite symmetric matrix, $\langle \cdot, \cdot \rangle$ is the usual scalar product in \mathbb{R}^2 and $b \in \mathbb{R}^2$ is fixed such that $\langle A^{-1}b, b \rangle < 1$; here A^{-1} denotes the inverse of A . The pair (\mathbb{R}^2, F_b) is a Randers-type Minkowski plane which describes the anisotropic Luneburg-type refraction in optical crystals or the electromagnetic field of the physical space-time in general relativity (in higher dimension), see Randers [27]. Note that (\mathbb{R}^2, F_b) is reversible if and only if $b = (0, 0)$.

Let $R, r > 0$ and $x, y \in \mathbb{R}^2$ be arbitrarily fixed. Since the forward and backward indicatrices $I^+(x, r) = \partial B^+(x, r)$ and $I^-(y, R) = \partial B^-(y, R)$ are ellipses which can be obtained from each other by translation and dilation, equality in the Brunn-Minkowski inequality (4.24) holds for any choice of $A = B^+(x, r)$ and $B = B^-(y, R)$ in (\mathbb{R}^2, F_b) , due to Corollary 4.4; see also Figure 1(a).

(b) (*Matsumoto mountain slope metric*) Let $F_\alpha : T\mathbb{R}^2 \rightarrow [0, \infty)$ be defined by

$$F_\alpha(x, y) := F_\alpha(y) = \begin{cases} \frac{y_1^2 + y_2^2}{v\sqrt{y_1^2 + y_2^2 + \frac{g}{2}y_2 \sin \alpha}}, & y = (y_1, y_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}; \\ 0, & y = (y_1, y_2) = (0, 0), \end{cases} \quad (4.26)$$

where $\alpha \in [0, \pi/2)$, $v > 0$ and $g \approx 9.81$. If we assume that $g \sin \alpha < v$, it turns out that (\mathbb{R}^2, F_α) is a Minkowski plane, describing the law of walking with a constant speed $v[m/s]$ under the effect of gravity on a mountain slope having the angle α w.r.t. the horizontal plane, see Matsumoto [22]. It is clear that (\mathbb{R}^2, F_α) is reversible if and only if $\alpha = 0$, which corresponds to the Euclidean setting and F_b reduces to the standard (reversible) metric $F_0(y_1, y_2) = \sqrt{y_1^2 + y_2^2}/v$.

Let $x, y \in \mathbb{R}^n$ and $r, R > 0$ be arbitrarily fixed. We notice that the indicatrices $I^+(x, r) = \partial A = \{z \in \mathbb{R}^2 : F_\alpha(z - x) = r\}$ and $I^-(y, R) = \partial B = \{z \in \mathbb{R}^2 : F_\alpha(y - z) = R\}$ are convex limaçons which

cannot be obtained from each other by dilation and translation, unless $\alpha = 0$ (i.e., the mountain slope vanishes), see also Figure 1(b). Thus, due to Corollary 4.4, for $\alpha \neq 0$ (i.e., we are in the non-Euclidean setting) any choice of $A = B^+(x, r)$ and $B = B^-(y, R)$ in (\mathbb{R}^2, F_α) provides strict inequality in the Brunn-Minkowski inequality (4.24).

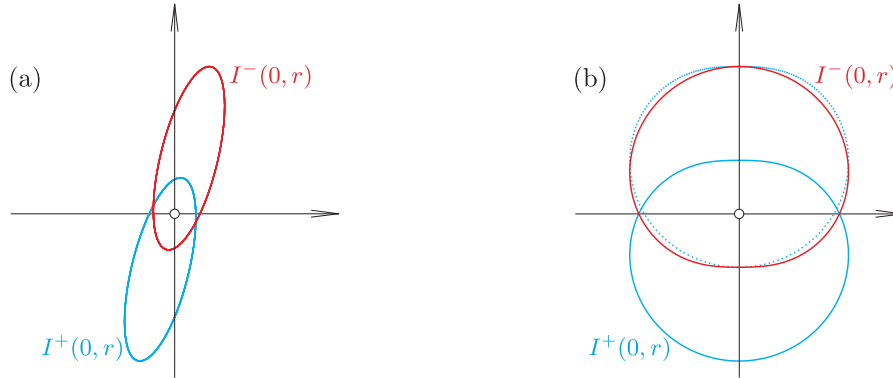


FIGURE 1. (a) Let $A = [5, -1; -1, 1]$ and $b = (1/5, 1/2)$ in Example 4.2(a). The forward and backward balls (which are ellipses) with the same radius can be translated to each other. (b) Let $\alpha \approx 35^\circ$ and $v = 6$ in Example 4.2(b). The forward and backward balls with the same radius cannot be translated to each other.

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